Design of Robust Dynamic Average Consensus Estimators

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***Abstract*— The block diagram for a general average consensus estimator is developed and we show how this can be used to easily identify properties of the estimator. This structure is then used to design average consensus estimators which achieve exact average consensus for constant inputs, are robust to initial conditions and switching graph topologies, and are internally stable. Additionally, the estimators have the optimal worst-case asymptotic convergence rate over the set of connected undirected graphs whose weighted Laplacian matrices have nonzero eigenvalues in a known interval $[\lambda_{\min}, \lambda_{\max}]$.**

Two designs are presented. The first is a modification of the polynomial filter estimator proposed by Kokiopoulou and Frossard [1] which is the optimal estimator having only one state variable. The proposed design is robust to initial conditions, but not robust to switching graph topologies. The second design uses root locus techniques to obtain higher-dimensional estimators in closed-form which are robust to both initial conditions and switching graph topologies. Plots of the worst-case asymptotic convergence factor of each estimator are given as a function of the ratio $\lambda_{\min}/\lambda_{\max}$.

I. INTRODUCTION

We consider the dynamic average consensus problem where each agent in a network uses communication with its network neighbors along with a local estimator to calculate the average input of all the agents [2], [3], [4], [5], [6]. Many applications in the decentralized control of multi-agent systems require average consensus estimators which are reliable and converge quickly.

In this paper we design average consensus estimators which have zero steady-state error for constant inputs even when the underlying communication network changes (such as the addition/removal of agents or dropped packets). Additionally, the estimators are designed to converge quickly for a large set of communication graphs characterized by their connectivity. A block diagram for a general estimator is developed and simple conditions on the block diagram determine properties of the estimator such as exactness, internal stability, robustness to initial conditions, and robustness to switching graph topologies.

We show that the worst-case asymptotic convergence factor is a monotonically non-increasing function of the ratio $\lambda_{\min}/\lambda_{\max}$ where $\lambda_{\min}$ and $\lambda_{\max}$ are bounds on the nonzero eigenvalues of the weighted Laplacian. Therefore, the design of optimal estimators is composed of two problems: 1) choose the edge weights in a distributed manner to maximize $\lambda_{\min}/\lambda_{\max}$, and 2) design the estimator to guarantee the desired properties for any connected undirected weighted graph having Laplacian eigenvalues in the interval $[\lambda_{\min}, \lambda_{\max}]$. In this paper we focus on a version of the second problem: given $\lambda_{\min}$ and $\lambda_{\max}$, design an estimator which provides exact average consensus for constant inputs, is internally stable, is robust to initial conditions and switching graph topologies, and optimizes the worst-case asymptotic convergence rate.

Two design approaches are given. The first design is a modification of the polynomial filter estimator proposed by Kokiopoulou and Frossard [1]. The estimator in [1] is shown not to be robust to initial conditions, but a simple modification of the block diagram produces an estimator which is robust to initial conditions. Both estimators, however, are not robust to switching graph topologies. To overcome this difficulty, we present a different design which uses root locus methods to obtain higher-dimensional estimators which are robust to both initial conditions and switching graph topologies. The cost of having both properties, however, is a slower convergence rate. Plots of the worst-case asymptotic convergence factor normalized by the amount of communication required to implement the estimator are given for each estimator as a function of the ratio $\lambda_{\min}/\lambda_{\max}$.

Much research has been done on optimizing the convergence rate of average consensus estimators [7], [8]. However, most work focuses on the static consensus problem in which the inputs appear only through the initial conditions. Although any static consensus estimator can be easily converted to a dynamic one in which the inputs appear in the state update equations themselves, the resulting estimators are inherently not robust to initialization errors [9]. Elwin et al. [10] optimize the worst-case performance of robust dynamic average consensus estimators by applying numerical global optimization solvers, but no guarantees are given of finding the global optimum. Two-dimensional estimators are designed in [11] which are robust to both initial conditions and switching network topologies and are guaranteed to be globally optimal, but closed-form solutions are not given and the design method is not suitable for higher-dimensional estimators.

The rest of the paper is organized as follows. Section II sets up the average consensus problem. Section III gives the block diagram structure of an average consensus estimator and shows how the properties of the estimator can easily be determined from the diagram. Sections IV and V give two estimator designs; the first uses the polynomial filter approach and the second uses root locus techniques. The designs are compared in Section VI, and conclusions are given in Section VII.

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II. DYNAMIC AVERAGE CONSENSUS

Consider a group of N agents, each having a local scalar input signal u_i. Each agent runs an estimator which takes in its local input along with information from its neighbors to produce a local scalar output signal y_i. The goal of each estimator is to have its output y_i track the global average of all the local inputs. We model the communication topology as a weighted undirected graph G. Define the adjacency matrix of G to be $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ where $a_{ij} = a_{ji} > 0$ if agents i and j can communicate and zero otherwise (with $a_{ii} = 0$). The neighbors of agent i, denoted $\mathcal{N}_i$, is the set of agents with which agent i can communicate. The degree of agent i, denoted $\deg(i)$, is the number of agents in $\mathcal{N}_i$. Define the $N\times 1$ vectors 1_N and 0_N of all ones and zeros, respectively, and I_N as the $N \times N$ identity matrix. Then the Laplacian matrix is $L = \text{diag}(A1_N) - A$ which is positive semidefinite and satisfies $Lv = 0_N$ where $v = 1_N/\sqrt{N}$. The algebraic connectivity of the graph is the second smallest eigenvalue of L, denoted λ_{min}. The graph is connected if and only if $\lambda_{\text{min}} > 0$.

If the graph topology is known, then the weights a_{ij} can be chosen to optimize system performance [12]. When the graph is unknown, however, it is often useful to choose a weighting scheme which bounds the eigenvalues of the Laplacian. For example, the decentralized weighting scheme $a_{ij} = 1/[\deg(i) + \deg(j)]$ restricts the eigenvalues of L to the interval $[0, 1]$ which allows us to use $\lambda_{\text{max}} = 1$ when the graph is unknown [9].

We now define several properties of average consensus estimators. For simplicity, we assume we have a single estimator design, and that each agent runs a local copy of this estimator. We stack the local scalar inputs u_i and outputs y_i into vectors u and y, each of which can depend on the discrete time variable k.

Definition 1 (Exact): An estimator is said to achieve *exact* average consensus when for any constant input u, the output $y(k)$ converges to $\frac{1}{N}1_N\sum_{i=1}^{N} u_i$ as $k \to \infty$.

Definition 2 (Internally stable): An estimator is said to be *internally stable* when for any initial internal states and any bounded inputs $u(\cdot)$, all internal states remain bounded in forward time.

Definition 3 (Robust to initial conditions): An estimator is said to be *robust to initial conditions* when the limit of $y(k)$ as $k \to \infty$ does not depend on the initial internal states.

Definition 4 (Ergodic): Suppose the input u is constant, and suppose the weighted communication graph changes at each time step according to some random process. Here we assume only that the *expected* graph is connected and undirected; the graph at each time step need not be connected nor satisfy the above symmetry assumption $a_{ij} = a_{ji}$. Let L_k denote the resulting Laplacian at time k, and assume L_k is i.i.d. and independent of the input and any initial states. Then we call an estimator *ergodic* when for any sufficiently small nonzero variance of the Laplacian process L_k, the output process $y(k)$ is asymptotically mean ergodic, meaning that its time average converges to its statistical average as $k \to \infty$.

For one-hop estimators in which agents communicate only with their one-hop neighbors at each time step, the statistical average of the output $y(k)$ is the output of the estimator when the switching Laplacian L_k is replaced by its expected value. For multi-hop estimators in which higher powers of the Laplacian appear, the statistical average of the output $y(k)$ is the output of the estimator when each power of L_k is replaced by its expected value. If an ergodic estimator is exact under these expected Laplacian powers, then under the switching Laplacian a local low-pass filter can be applied to each local output to obtain the exact global average. Conditions for an estimator to be ergodic are given in [13].

These properties of the estimator may depend on the graph; for example, typically an estimator can be exact only for connected graphs, and typically an estimator can be internally stable only for graphs whose Laplacian eigenvalues satisfy a known upper bound. We now state the problem to be solved in this paper.

Problem 1: Given λ_{min} and λ_{max} with $0 < \lambda_{\text{min}} \le \lambda_{\text{max}}$, design an estimator which is exact, internally stable, robust to initial conditions, and ergodic for all connected undirected weighted graphs whose nonzero Laplacian eigenvalues lie in the interval $[\lambda_{\text{min}}, \lambda_{\text{max}}]$. Furthermore, the estimator should have the optimal worst-case asymptotic convergence rate over all such graphs.

III. BLOCK DIAGRAM

To design average consensus estimators which solve Problem 1, we first characterize the block diagram of average consensus estimators and show how the properties of the estimator relate to the block diagram. Bai et al. [6] give the block diagram for the generalized PI estimator in which each agent has n internal state variables and two variables are communicated with neighboring agents at each iteration. We generalize this to estimators which communicate a general number of variables to neighboring agents at each iteration.

The block diagram of the estimator used throughout this paper is shown in Figure 1a which shows the full system including all agents. The transfer functions are given by

$$\begin{aligned} H_\ell(z) &= h_\ell(z) I_N, \qquad && \ell = 1, \dots, r \\ G_\ell &= g_\ell I_N, && \ell = 1, \dots, r-1 \end{aligned}$$

where $h_\ell(z)$ and g_ℓ are implemented on each agent, g_ℓ are constant gains, and r is the number of L blocks. The number of internal state variables on each agent, n, is equal to the sum of the number of states in $h_\ell(z)$ for $\ell = 1, \dots, r$. If either $n = 1$ or $h_\ell(z)$ is strictly proper for $\ell = 1, \dots, r$, then the estimator can be implemented with r variables transmitted to one-hop neighbors at each iteration. Multiplying a signal x by L is implemented on each agent by taking a weighted average of the difference between neighbors as follows,

$$(Lx)_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_i - x_j).$$

To do the design, we use the separated system shown in Figure 1b. For undirected graphs, the Laplacian is symmetric and can be diagonalized as $D = Q^T L Q$ where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_N)$, $Q = \begin{bmatrix} q_1 & \ldots & q_N \end{bmatrix} \in \mathbb{R}^{N \times N}$ is orthogonal, and λ_m are the eigenvalues of L. Without loss of generality, we can assume $\lambda_1 = 0$ and $q_1 = 1_N/\sqrt{N}$. To get the separated system, we multiply the input $u(z)$ by q_m^T so that the output is $q_m^T y(z)$. The output of the full system can then be recovered using

$$y(z) = Q^T Q y(z) = q_1[q_1^T y(z)] + \sum_{m=2}^{N} q_m[q_m^T y(z)] \quad (1)$$

which is the sum of the output of the separated system in the consensus direction ($m = 1$) and the disagreement directions ($m = 2, \ldots, N$).

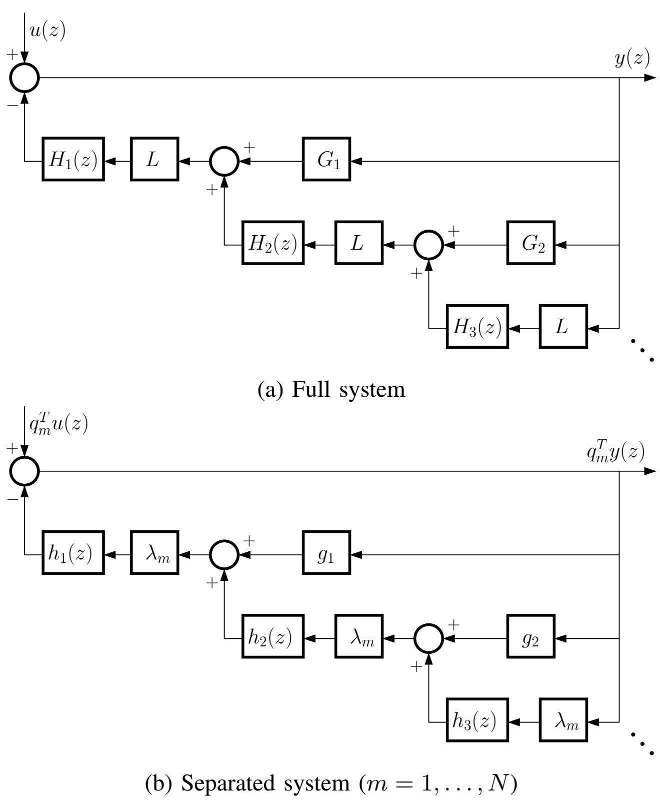


Fig. 1: Block diagram of an average consensus estimator. The diagram is shown with $r = 3$ communication states, but the ellipses indicate how to generalize the diagram for general r.

Lemma 1: The transfer function of the separated system in Figure 1b is

$$\frac{q_m^T y(z)}{q_m^T u(z)} = \frac{1}{F(z, \lambda_m)} \quad (2)$$

where

$$F(z, \lambda_m) = 1 + \sum_{\ell=1}^{r} f_\ell(z) \lambda_m^\ell \quad (3)$$

$$f_\ell(z) = g_\ell \prod_{j=1}^{\ell} h_j(z), \quad \ell = 1, \ldots, r \quad (4)$$

with $g_r = 1$.

We make the following assumption throughout the paper which excludes algebraic loops in Figure 1.

Assumption 1: The transfer functions $f_\ell(z)$ for $\ell = 1, \ldots, r$ are either strictly proper or identically zero.

The following theorem characterizes the properties of the estimator based on the structure of the block diagram.

Theorem 1: Given a Laplacian matrix corresponding to a connected undirected weighted graph, assume that transfer function (2) has all poles strictly inside the unit circle for all $\lambda_m \in \operatorname{eig}(L) \backslash \{0\}$. Then the estimator in Figure 1 is

1) exact if and only if $h_\ell(z)$ has a pole at $z = 1$ for some $\ell = 1, \ldots, r$,
2) internally stable if and only if for all $\ell = 1, ...r$, $h_\ell(z)$ has no poles strictly outside the unit circle and no repeated poles on the unit circle,
3) robust to initial conditions if and only if $h_1(z)$ is stable, and
4) ergodic if robust to initial conditions and $h_1(z)$ is strictly proper.

Proof: The proof, omitted for brevity, relies on analyzing the separated system in the consensus direction ($\lambda_1 = 0$) and the disagreement directions ($\lambda_m > 0$ for $m = 2, \ldots N$) and combining the results using equation (1). ■

Theorem 1 shows that the properties of the estimator depend on how the model of the input appears in the block diagram. Specifically, we have the following properties as shown in Figure 2:

- To be exact, the estimator must contain an integrator.
- To be internally stable, the output must pass through the Laplacian before reaching any integrator.
- To be robust to initial conditions, any integrator states must pass through the Laplacian before reaching the output.
- To be ergodic, it is sufficient that the estimator is robust to initial conditions and any state passed through the Laplacian is filtered by a strictly proper transfer function before reaching the output.

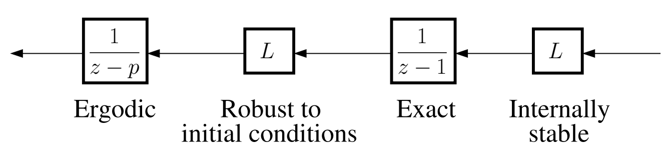


Fig. 2: Properties of the estimator based on the structure of the block diagram.

Definition 5: The *worst-case asymptotic converge factor* of the estimator in Figure 1 is defined as

$$\alpha = \max\{\alpha_1, \alpha_2\} \quad (5)$$

where

$$\alpha_1 = \max_{\lambda\in[\lambda_{\min},\lambda_{\max}],z\in\mathbb{C}} |z| \quad \text{subject to } 0 = F(z,\lambda) \qquad (6)$$

$$\alpha_2 = \max_{z\in\mathbb{C}} |z| \quad \text{subject to } 0 = d(z). \qquad (7)$$

and $h_1(z) = n(z)/d(z)$.

Definition 6: Given the estimator in Figure 1 which requires K variables to be transmitted to neighboring agents during each iteration, the *normalized worst-case asymptotic converge factor* is defined as $\tilde{\alpha} = \sqrt[K]{\alpha}$.

Lemma 2: The worst-case asymptotic convergence factor α is a monotonically non-increasing function of the ratio $\lambda_{\min}/\lambda_{\max}$.

Proof: Omitted for brevity. ■

Note that this paper focuses on exactness for constant inputs. The case for time-varying inputs is similar where the integrator is replaced by the model of the input; see [6] for details.

IV. POLYNOMIAL FILTER DESIGN

One design method proposed by Kokiopoulou and Frossard [1] uses a polynomial filter to shape the spectrum of the Laplacian. The design in [1] is given as a static estimator. An equivalent dynamic estimator, however, is not robust to initial conditions. Using the block diagram of the estimator, we propose a simple modification to make the estimator robust to initial conditions.

The polynomial filter design applies a polynomial p_r of degree r to the Laplacian to minimize the convergence factor. The static polynomial filter algorithm is given by

$$x_{k+1} = p_r(L)x_k$$

where $x_0 = u$ and

$$p_r(L) = I + p_1L\Big(I + p_2L\big[I + p_3L(\ldots)\big]\Big).$$

Note that the standard static consensus algorithm uses the choice $p_1(L) = I - L$. Using the separated system, this may be written as a scalar polynomial in λ,

$$p_r(\lambda) = 1 + p_1\lambda\Big(1 + p_2\lambda\big[1 + p_3\lambda(\ldots)\big]\Big),$$

where $\lambda \in \operatorname{eig}(L)$. The characteristic polynomial of the separated system has a single root at $p_r(\lambda)$. Figure 3a gives the block diagram of an equivalent dynamic estimator with the same characteristic polynomial. The estimator is not robust to initial conditions since the integrator state does not pass through the Laplacian before reaching the output. To make the estimator robust to initial conditions, we add a Laplacian block after the integrator as shown in Figure 3b.

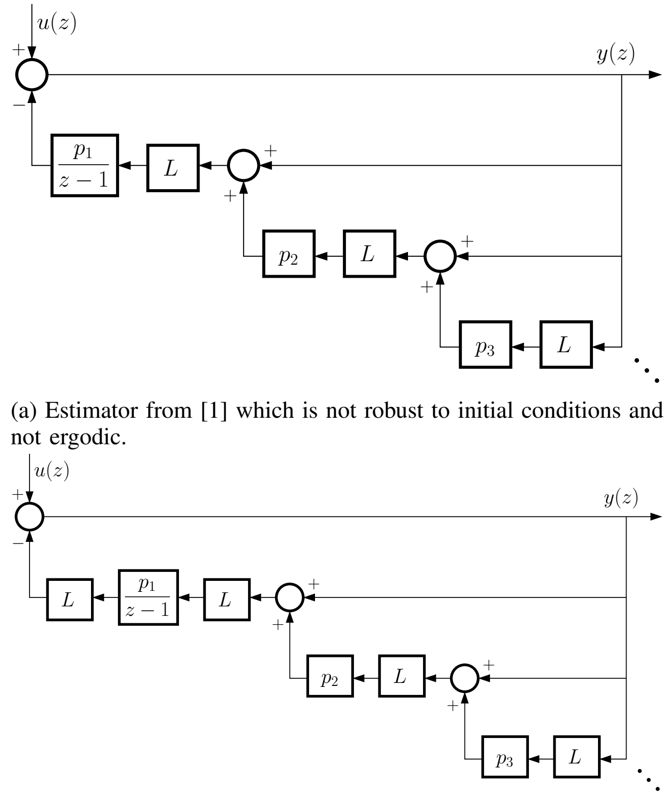
The polynomial filtering problem can be stated as follows.

Problem 2: Given $\lambda_{\min}$, $\lambda_{\max}$, and r, solve

$$\alpha = \min_{p_r} \max_{\lambda\in[\lambda_{\min},\lambda_{\max}]} |p_r(\lambda)|$$

where $p_r(\lambda)$ is a polynomial of degree r of the form

$$p_r(\lambda) = \begin{cases} 1 + p_1\lambda\Big(1 + p_2\lambda\big[1 + p_3\lambda(\ldots)\big]\Big), & \text{Figure 3a} \\ 1 + p_1\lambda^2\Big(1 + p_2\lambda\big[1 + p_3\lambda(\ldots)\big]\Big), & \text{Figure 3b.} \end{cases}$$



(a) Estimator from [1] which is not robust to initial conditions and not ergodic.

(b) Proposed estimator which is robust to initial conditions, but not ergodic.

Fig. 3: Block diagram of polynomial filter estimators.

To implement the polynomial filter estimator, the state must be averaged with neighboring agents r times with each result stored locally on each agent. After r iterations, each agent then applies the polynomial p_r to the stored values and updates its current estimate (see [1] for details). Therefore, r consensus iterations are needed in order to perform a single update iteration, so the optimal normalized worst-case asymptotic convergence factor can be found by solving the following problem.

Problem 3: Given $\lambda_{\min}$ and $\lambda_{\max}$, solve

$$\tilde{\alpha} = \min_r \sqrt[r]{\alpha}$$

where α is the solution to Problem 2.

The solution to the non-robust version of Problem 2 is shown in [14] to be given by shifted and scaled Chebyshev polynomials of the first-kind. Furthermore, the solution to Problem 3 for both the robust and non-robust estimators is $\tilde{\alpha} = (1-\sqrt{\lambda_{\min}/\lambda_{\max}})/(1+\sqrt{\lambda_{\min}/\lambda_{\max}})$ which is achieved in the limit as $r \to \infty$. This is plotted as a function of the ratio $\lambda_{\min}/\lambda_{\max}$ in Figure 6.

Neither estimator in Figure 3 is ergodic. The estimator in Figure 3a is not robust to initial conditions and therefore not ergodic. Theorem 1 contains only a sufficient condition for ergodicity, so ergodicity of the estimator in Figure 3b cannot be proven since $h_1(z) = 1$ is not strictly proper. Simulations indicate, however, that the estimator is not ergodic. In the next section, higher-dimensional estimators are designed which are both robust to initial conditions and ergodic.

V. ROOT LOCUS DESIGN

From Section III, we see that the closed-loop poles of the estimator in Figure 1 are the solutions to $0 = F(z, \lambda)$ where $F(z, \lambda)$ is given by equation 3. To simplify the design, we assume $r = 2$ with $h_1(z)$ and $h_2(z)$ strictly proper and $g_1 = 1$. The estimator can then be implemented using only two transmission variables, so the normalized worst-case asymptotic convergence rate is $\tilde{\alpha} = \sqrt{\alpha}$. Equation (3) then becomes

$$F(z, \lambda) = 1 + \lambda h_1(z) \left[1 + \lambda h_2(z)\right]. \tag{8}$$

For the estimator to be exact, internally stable, robust to initial conditions, and ergodic, we need $h_1(z)$ to be stable, $h_2(z)$ to have a pole at $z = 1$, and $h_2(z)$ to have no poles strictly outside the unit circle or repeated poles on the unit circle. To solve Problem 1, we need to design $h_1(z)$ and $h_2(z)$ with the given restrictions such that α is minimized where α is given by equation (5).

Solving $0 = F(z, \lambda)$ is a quadratic root locus problem which has been studied in [15]. However, instead of viewing the problem as a quadratic root locus, we do the design as two nested linear root locus problems. For fixed $\bar{\lambda}$, the closed-loop poles of the system are on the h_1-locus

$$0 = 1 + \lambda h_1(z) \left[1 + \bar{\lambda} h_2(z)\right] \qquad (h_1\text{-locus})$$

when $\lambda = \bar{\lambda}$. To design the h_1-locus, we need the roots of

$$0 = 1 + \bar{\lambda} h_2(z) \qquad (h_2\text{-locus})$$

which is a root locus in the parameter $\bar{\lambda}$. Using the decomposition $h_2(z) = n(z)/d(z)$, we see that

$$1 + \bar{\lambda} h_2(z) = \frac{d(z) + \bar{\lambda} n(z)}{d(z)} = \frac{\text{closed-loop poles}}{\text{open-loop poles}},$$

so the closed-loop poles of the h_2-locus become open-loop zeros in the h_1-locus, and open-loop poles of the h_2-locus remain open-loop poles in the h_1-locus.

To design the estimator, we first choose $h_2(z)$ such that the closed-loop poles are inside a circle of radius $\alpha < 1$ centered at the origin of the complex plain, keeping in mind that the closed-loop poles become the open-loop zeros of the h_1-locus. Then we choose $h_1(z)$ such that the h_1-locus remains inside the α-circle for all $\lambda \in [\lambda_{\min}, \lambda_{\max}]$. Conditions from locations on the root loci are used to solve for α and the gains of $h_1(z)$ and $h_2(z)$. Then the estimator is the solution to Problem 1 where α is the worst-case asymptotic convergence factor.

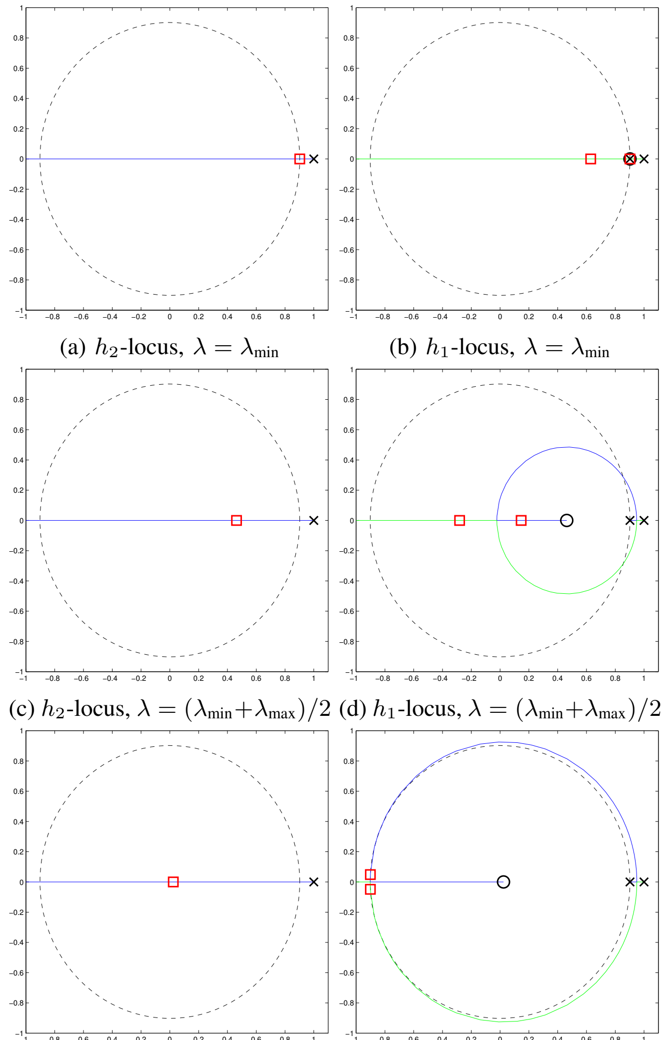
We will need the following definitions. The discriminant of $F(z, \lambda)$ with respect to λ is

$$r(z) = h_1(z) \left[h_1(z) - 4 h_2(z)\right]. \tag{9}$$

Also, define the ratio $\lambda_0 = \lambda_{\min}/\lambda_{\max}$ and define the coefficients of $F(z, \lambda_{\max})$ as

$$F(z, \lambda_{\max}) = \sum_{j=0}^{n} c_j z^j. \tag{10}$$

We now use the developed root locus technique to solve Problem 1 for $n = 2$ and $n = 4$.



(a) h_2-locus, $\lambda = \lambda_{\min}$ (b) h_1-locus, $\lambda = \lambda_{\min}$ (c) h_2-locus, $\lambda = (\lambda_{\min}+\lambda_{\max})/2$ (d) h_1-locus, $\lambda = (\lambda_{\min}+\lambda_{\max})/2$ (e) h_2-locus, $\lambda = \lambda_{\max}$ (f) h_1-locus, $\lambda = \lambda_{\max}$

Fig. 4: Root locus plots for $n = 2$ with $\lambda_0 = 0.1$ (small $\lambda_{\min}/\lambda_{\max}$ case). The dashed circle has radius α. $\times$'s are open-loop poles, $\bigcirc$'s are open-loop zeros, and $\square$'s are closed-loop poles at the given value of λ.

A. Case: $n = 2$

We can parameterize $h_1(z)$ and $h_2(z)$ as

$$h_1(z) = \frac{k_p}{z - \gamma}, \qquad h_2(z) = \frac{k_I}{z - 1}.$$

Since the poles of $h_1(z)$ must lie in the α-circle, we must have $\gamma \in [-\alpha, \alpha]$. To design the h_2-locus, we choose the gain k_I such that the pole is at $z = \alpha$ when $\bar{\lambda} = \lambda_{\min}$ as shown in Figure 4a. Therefore, we have

$$0 = 1 + \lambda_{\min} h_2(\alpha) \tag{11}$$

which gives

$$k_I = \frac{1 - \alpha}{\lambda_{\min}}.$$

The h_1-locus has relative degree one, so the closed-loop pole approaches $z = -\infty$ along the negative real axis as $\lambda \to \infty$. To stabilize the pole for the largest ratio $\lambda_{\text{min}}/\lambda_{\text{max}}$, we choose $\gamma = \alpha$. Two conditions are needed to solve for k_p and α. The root loci are shown for small λ_0 in Figure 4. Due to the shape of the locus, the closed-loop poles can only exit the α-circle at $z = -\alpha$ or when the closed-loop poles are complex conjugates with magnitude α. From Vieta's formulas, we use $c_0/c_2 = \alpha^2$ to force the closed-loop poles to be on the α-circle when $\lambda = \lambda_{\text{max}}$. For small λ_0, we use $0 = r(-\alpha)$ to prohibit the poles from crossing the point $z = -\alpha$ more than once. For large λ_0, however, the pole leaves the α-circle before $\lambda = \lambda_{\text{min}}$, so we force it back inside the α-circle at $\lambda = \lambda_{\text{min}}$ using $0 = F(-\alpha, \lambda_{\text{min}})$. Therefore, the conditions are

$$0 = \alpha^2 - \frac{c_0}{c_2}$$
$$0 = \begin{cases} r(-\alpha), & \lambda_0 \text{ small} \\ F(-\alpha, \lambda_{\text{min}}), & \lambda_0 \text{ large}. \end{cases}$$

We can solve the conditions for both the small and large λ_0 cases, and then use the condition $0 = r(-\alpha) = F(-\alpha, \lambda_{\text{min}})$ to find the transition point between the two solutions. This gives

$$k_p = \frac{1}{\lambda_{\text{max}}} \frac{\alpha(1-\alpha)\lambda_0}{\alpha + \lambda_0 - 1}$$
$$\alpha = \begin{cases} \alpha_0, & 0 < \lambda_0 \leq 3 - \sqrt{5} \\ \alpha_1, & 3 - \sqrt{5} < \lambda_0 \leq 1 \end{cases}$$

where

$$\alpha_0 = \frac{\lambda_0^2 - 8\lambda_0 + 8}{8 - \lambda_0^2}$$
$$\alpha_1 = \frac{\sqrt{(1-\lambda_0)(5\lambda_0^2 - \lambda_0^3 + 4)} - \lambda_0 + \lambda_0^2}{2(\lambda_0^2 + 1)}.$$

Note that the global optimum for the case $n = 2$ was solved numerically in [11]. The root locus design procedure, however, gives more insight along with giving a closed-form solution. By comparing solutions, we see that the root locus design gives the same solution as that of [11], and is therefore the global optimum.

B. *Case:* $n = 4$

We can parameterize $h_1(z)$ and $h_2(z)$ as

$$h_1(z) = k_p \frac{(z - \eta_1)}{(z - \gamma_1)(z - \gamma_2)}, \quad h_2(z) = k_I \frac{(z - \eta_2)}{(z - 1)(z - \gamma_3)}.$$

Using the choices $\gamma_1 = \gamma_2 = \alpha$, $\gamma_3 = \alpha^2$, and $\eta_1 = \eta_2 = 0$ gives the root loci in Figure 5. From the h_2-locus, we choose the gain k_I using equation (11) which gives

$$k_I = \frac{(1-\alpha)^2}{\lambda_{\text{min}}}.$$

The conditions used to solve for k_p and α are similar to those used in the $n = 2$ case, except for the condition at $\lambda = \lambda_{\text{max}}$. Instead of forcing complex conjugate roots on the α-circle, we now force there to be two double roots using the discriminant of the quartic equation (10) (see [16]). This occurs when the two pairs of complex conjugate roots break apart on the α-circle. Therefore, the conditions are

$$0 = 64c_4^3 c_0 - 16c_4^2 c_2^2 + 16c_4 c_2^2 c_3 - 16c_4^2 c_3 c_1 - 3c_2^4$$
$$0 = \begin{cases} r(-\alpha), & \lambda_0 \text{ small} \\ x(-\alpha, \lambda_{\text{min}}), & \lambda_0 \text{ large} \end{cases}$$

which gives

$$k_p = \left(2 - \lambda_0 + 2\sqrt{1 - \lambda_0}\right) k_I$$
$$\alpha = \begin{cases} \alpha_0, & 0 < \lambda_0 \leq 2(\sqrt{2} - 1) \\ \alpha_1, & 2(\sqrt{2} - 1) < \lambda_0 \leq 1 \end{cases}$$

with

$$\alpha_0 = \frac{2 - \beta + 4(1 - \sqrt{2 - \beta})}{2 + \beta}$$
$$\alpha_1 = \frac{1 + \beta + 2(1 - \sqrt{2 + \beta})}{1 + \beta}$$

where $\beta = 2\sqrt{1 - \lambda_0} - \lambda_0$. For the $n = 4$ case, the given solution is not proven to be globally optimal. However, we conjecture that the proposed solution is the global optimum. The normalized worst-case spectral radius of the estimator, $\sqrt{\alpha}$, is plotted in Figure 6 as a function of $\lambda_0 = \lambda_{\text{min}}/\lambda_{\text{max}}$.

VI. COMPARISON

The normalized worst-case asymptotic convergence factor for each estimator is shown in Figure 6. Both polynomial filters in Figure 3 have the fastest normalized convergence rate for all $\lambda_{\text{min}}/\lambda_{\text{max}}$, but in each case the rate shown is only achieved in the limit as the number of consensus iterations per time step approaches infinity ($r \to \infty$). In practice, only a finite number of consensus iterations can be used at each time step resulting in slower convergence. Also, both polynomial filter estimators are not ergodic. The root locus designs converge slower in general, but are both ergodic. This shows the trade-off between convergence rate and ergodicity.

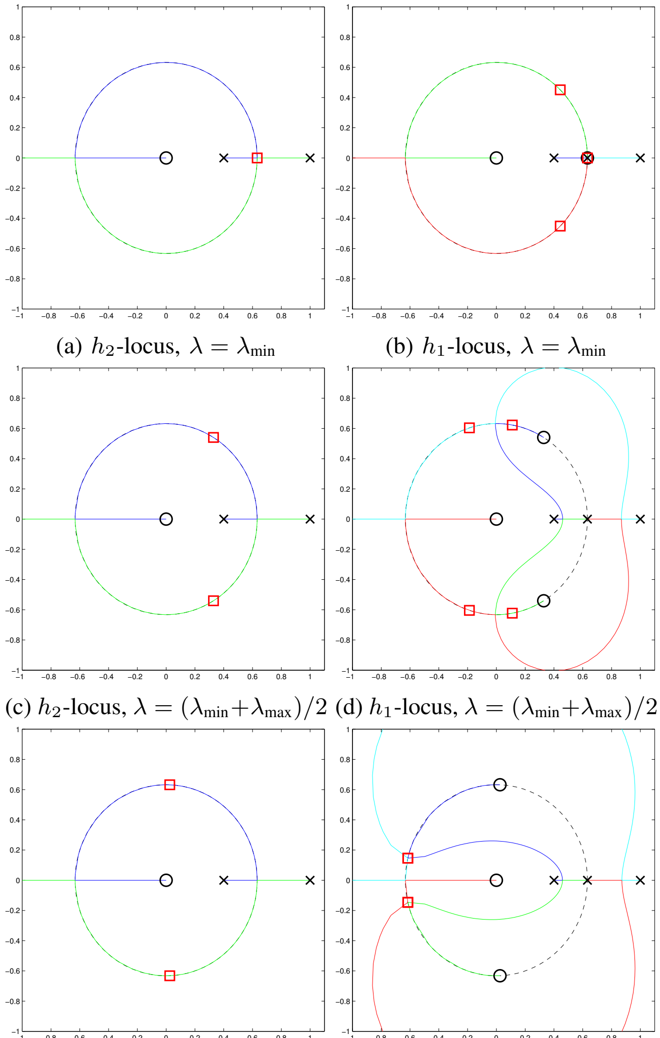
From Lemma 2, we have that α is monotonically non-increasing. Therefore, the general design process is as follows:

1) Design the Laplacian to maximize $\lambda_{\text{min}}/\lambda_{\text{max}}$,
2) Design the estimator to have the desired properties.

The first problem can be solved exactly using semidefinite programming if the graph is known and symmetric [7], otherwise a weighting scheme can be chosen such as inverse out-degree weighting [9]. The second problem can be solved using Figure 6 to choose the estimator based on the desired worst-case asymptotic convergence factor and other properties such as robustness to intial conditions and ergodicity.

VII. CONCLUSIONS

We proposed two design methods for constructing average consensus estimators with optimal worst-case asymptotic convergence rate that also have other desired properties. The estimator properties are characterized using the structure of



(a) h_2-locus, $\lambda = \lambda_{\text{min}}$ (b) h_1-locus, $\lambda = \lambda_{\text{min}}$

(c) h_2-locus, $\lambda = (\lambda_{\text{min}}+\lambda_{\text{max}})/2$ (d) h_1-locus, $\lambda = (\lambda_{\text{min}}+\lambda_{\text{max}})/2$

(e) h_2-locus, $\lambda = \lambda_{\text{max}}$ (f) h_1-locus, $\lambda = \lambda_{\text{max}}$

Fig. 5: Root locus plots for $n = 4$ with $\lambda_0 = 0.1$ (small λ_0 case). The dashed circle has radius α. $\times$'s are open-loop poles, ◯'s are open-loop zeros, and □'s are closed-loop poles at the given value of λ.

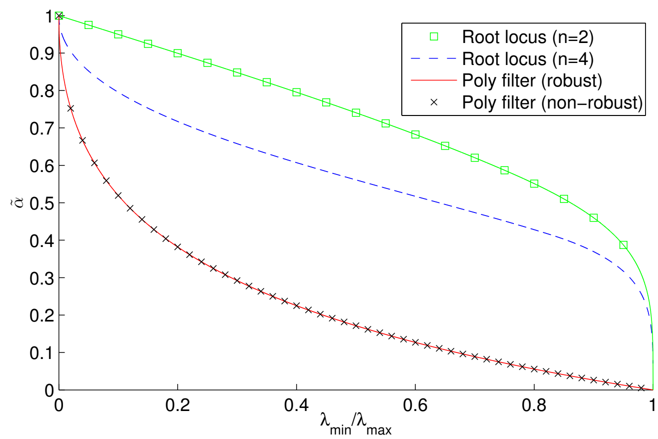


Fig. 6: Normalized worst-case convergence factor ($\tilde{\alpha}$) as a function of $\lambda_{\text{min}}/\lambda_{\text{max}}$ for the root locus estimators from Section V with $n = 2$ and $n = 4$ and the polynomial filter estimators from Section IV (robust to initial conditions and not robust to initial conditions, respectively).

the block diagram. The one-dimensional polynomial filter estimator from [1] was shown not to be robust to initial conditions. By modifying the block diagram, a similar estimator was proposed which is robust to initial conditions. Both polynomial filters, however, are not ergodic. To obtain both robustness to initial conditions and ergodicity, estimators of dimension two and four were designed using root locus techniques which achieve exact average consensus quickly and are internally stable, robust to initial conditions, and ergodic. The $n = 4$ design is the fastest linear discrete-time dynamic average consensus estimator we know of which has all four desired properties.

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