Feedback Optimization of Nonlinear Dynamical Systems in Time-Varying Environments via Output Regulation

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***Abstract*— Feedback optimization has emerged as a promising approach for regulating dynamical systems to optimal steady states implicitly defined as solutions to an underlying optimization problem. Despite their popularity, existing methods face two key limitations: (i) reliable performance is restricted to time-invariant or slowly varying settings, and (ii) convergence rates are limited by the need for the controller to operate orders of magnitude slower than the plant. These limitations can be traced back to the reliance of existing techniques on numerical optimization methods. In this paper, we develop a new perspective on feedback optimization by casting it as an output regulation problem. Building on this connection, we introduce a design methodology that combines output-feedback stabilization with feedforward control. The resulting algorithms overcome limitations (i)–(ii), enabling reliable operation in time-varying environments without requiring timescale separations.**

I. Introduction

The framework of Feedback Optimization (FO) [1]–[5] is concerned with the problem of controlling dynamical systems to an optimal steady-state point, where optimality is implicitly defined by an underlying mathematical optimization problem. The foundational design approach underlying FO begins with an established optimization algorithm [6], and suitably adapts it to incorporate feedback measurements from the dynamical system in place of unknown quantities [7]. This modification results in a feedback loop between the optimization and the dynamical system. Although numerous FO methods have been developed, these techniques inherently carry over the limitations of the optimization algorithms from which they originate. Such limitations include bounds on the achievable convergence rate [8], limited robustness to uncertainty [9], and the inability to exactly track optimizers in time-varying settings [5], [9].

In this work, we approach the FO problem by showing that this is fundamentally an output regulation problem [10]–[12]. By establishing this connection, we are able to design novel FO algorithms inspired by the output regulation literature. The proposed approach overcomes two practical limitations of the existing literature: (i) our methods achieve exact (asymptotic) tracking in the face of time-varying disturbances (whereas existing techniques can only achieve approximate tracking [5]); and (ii) our approach removes the need for a timescale separation between the plant and the controller, thereby improving the achievable rate of convergence.

Related work. We study FO problems, in line with [1]–[5], [13], [14]. Recent works in FO explore exponential convergence rates [5], [9], sampled-data settings [15], constrained variants [16], nonconvex problems [17], gradient-free methods [18], [19], and approaches with no timescale separation [8]. Since the optimization problem studied in this work is inherently time-varying, it is closely related to the literature [20]. Of particular relevance are the recent works [21], [22], which establish the necessity of internal models in online optimization, as well as [23], which incorporates internal models as part of its algorithmic design.

An important element of this work is the reformulation of FO objectives as an output regulation problem, which links our work with the output regulation literature. This field originated in the 1970s with foundational work on linear systems [10], [11], [24], subsequently extended to nonlinear systems with both local [12] and global [25] results.

Contributions. This paper features three main contributions. First, we demonstrate how the classical FO problem can be formulated as an output regulation problem (see Definition 2). This reformulation enables the use of tools from the output regulation literature to address the FO problem. Second, we provide solvability results for the FO problem for plants subject to time-varying disturbances (see Theorem 1). To the best of our knowledge, this is the first solvability result for FO problems with time-varying disturbances, as most of the existing literature focuses on systems subject to constant disturbances. Third, we propose a novel design methodology to solve FO problems for plants subject to variable disturbances. Our design technique relies on a separation principle, which combines an observer to estimate unmeasurable dynamical states, and a static feedback control action. We prove local asymptotic convergence for these methods (see Theorem 2). Due to space limitations, some proofs are omitted here and presented in [26].

II. Problem formulation

We consider plants described by:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t), w(t)), \\ y(t) &= c(x(t), w(t)), \end{aligned} \tag{1}$$

where $t \in \mathbb{R}_{\geq 0}$ denotes time, $x(t) \in X \subseteq \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, $y(t) \in \mathbb{R}^q$ is the measurable output, and $w(t) \in W \subseteq \mathbb{R}^p$ is a disturbance. We assume that $f : X \times \mathbb{R}^m \times W \to \mathbb{R}^n$ and $c : X \times W \to \mathbb{R}^q$ are C^1.

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The signal $w(t)$ models exogenous disturbances affecting the plant and/or the measurement equations. Hence, we treat $w(t)$ as an unknown, unmeasurable, time-varying signal.

We study the problem of devising control actions that accomplish two goals: (i) locally stabilize the plant (1), and (ii) regulate (1) to an optimal equilibrium point. The second objective is formalized mathematically by:

$$\begin{aligned} &\underset{u\in\mathbb{R}^m, x\in X}{\text{minimize}} && \phi_0(u,x),\\ &\text{subject to:} && 0=f(x,u,w(t)), \end{aligned} \tag{2}$$

where $\phi_0 : \mathbb{R}^m \times X \to \mathbb{R}$. Problem (2) formalizes an equilibrium-selection task, seeking an equilibrium[1] input-state pair that minimizes the loss $\phi_0(u,x)$.

From an optimization standpoint, (2) is an optimization problem that is *parametrized* by the exogenous signal $w(t)$. This has two key implications: (i) because $w(t)$ is unknown and unmeasurable, solutions to (2) cannot be computed explicitly through a solver (since a numerical value for $w(t)$ cannot be substituted); and (ii) because $w(t)$ is a time-varying signal, (2) defines a *sequence* of optimization problems (one at each time t), and thus regulating the plant to solutions of (2) involves also *tracking these solutions over time.*

Problem 1 (***FO algorithm design***)**.** Construct a control algorithm such that: (i) the equilibrium of (1) with $w(t)\equiv 0$ is locally exponentially stable, and (ii) the states and inputs of (1) reach and track, with zero error, solutions of (2) as $t\to\infty$, for any time-varying $w(t)$ in some class. □

We refer to such a controller as a *feedback-optimization (FO) algorithm*, in line with [2], [5].

III. Feedback optimization as an output regulation problem

In this section, we show that, under suitable assumptions, Problem 1 can be recast as an output regulation problem.

Assumption 1 (***Properties of the objective***)**.** The map $(u,x)\mapsto\phi_0(u,x)$ is convex and differentiable, and $(u,x)\mapsto \nabla\phi_0(u,x)$ is Lipschitz continuous in $\mathbb{R}^m\times X$.

Convexity and smoothness are standard assumptions in optimization [20], which have been widely used [2], [5].

Assumption 2 (***Existence of a steady-state map***)**.** There exists a unique C^1 function $h:\mathbb{R}^m\times W\to X$ such that $f(h(u,w),u,w)=0$, for any $u\in\mathbb{R}^m$ and $w\in W$. □

Assumption 2 guarantees that, for constant inputs, (1) admits a steady-state operating point. This assumption holds in most practical cases of interest [27].

Using Assumption 2, problem (2) can be reformulated as:

$$\underset{u\in\mathbb{R}^m}{\text{minimize}} \quad \phi_0(u,h(u,w(t)))=\phi(u,w(t)), \tag{3}$$

where $\phi(u,w):=\phi_0(u,h(u,w))$.

[1]Our approach also applies to more general objectives than merely equilibrium selection (such as predictive control), provided that any state that satisfies the constraints can be written as a function of the control input and exogenous noise, as in Assumption 2.

Assumption 3 (***Temporal variability class***)**.** There exists a C^1 map $s:W\to\mathbb{R}^p$ such that $w(t)$ satisfies:

$$\dot w(t)=s(w(t)) \qquad \forall t\in\mathbb{R}_{\geq 0}. \tag{4}$$

Moreover, $w=0$ is an equilibrium of (4) and any trajectory is bounded. □

Assumption 3 excludes the presence of asymptotically stable modes; this is because, since these modes decay asymptotically, they do not affect the asymptotic solutions of (2). It also excludes the presence of unstable modes, as this would correspond to unbounded disturbance signals. Following established terminology, we call (4) the *exosystem.*

We also assume that W is a neighborhood of the origin in $\mathbb{R}^p$. This assumption is made without loss of generality: if the disturbance $w(t)$ evolves in a neighborhood of some $\bar w\neq 0$, a time-invariant change of variables $w\mapsto w-\bar w$ shifts the dynamics to a neighborhood of the origin without altering the optimizers of (2). We impose no restriction on the size of this neighborhood (which may be $W=\mathbb{R}^p$), and thus on the magnitude of $w(t)$ or on its temporal variation.

In stark contrast to classical FO approaches, which attain optimality only when the signal $w(t)$ is constant [2], [4], [5], our goal here is to leverage preliminary knowledge of the exosystem (4) to attain optimality for time-varying signals $w(t)$. Motivated by this, we introduce the following.

Definition 1 (***Critical trajectory***)**.** The function $u^\star:\mathbb{R}_{\geq 0}\to\mathbb{R}^m$ is said to be a critical trajectory of (3) if it satisfies $0=\nabla_u\phi(u^\star(t),w(t))$, $\forall t\in\mathbb{R}_{\geq 0}$. □

In words, a critical trajectory is an input signal that makes the gradient of the unconstrained problem vanish at all times.

Assumption 4 (***Existence and continuity of critical trajectories***)**.** The optimization (3) admits a critical trajectory. Moreover, every critical trajectory is continuous. □

Existence of a critical trajectory is guaranteed under mild assumptions (e.g., when $\phi_0(u,x)\to\infty$ as $|u|\to\infty$). Continuity of the critical trajectories also follows under standard conditions (e.g., continuity of $\phi_0(u,x)$).

We consider algorithms that rely solely on the plant's measurable output $y(t)$, and are described by an internal state $z(t)$, evolving on an open subset $Z\subseteq\mathbb{R}^{n_c}$ for some $n_c\in\mathbb{N}_{>0}$. Based on this information, the algorithm generates a control signal $u(t)\in\mathbb{R}^m$ at each time. Formally[2]:

$$\dot z(t)=F_c(z(t),y(t)), \qquad u(t)=G_c(z(t)), \tag{5}$$

where $F_c:Z\times\mathbb{R}^q\to\mathbb{R}^{n_c}$ and $G_c:Z\to\mathbb{R}^m$.

Problem 1 then entails designing the functions $F_c(z,y)$ and $G_c(z)$, together with n_c, such that the *gradient error* signal:

$$g(t)=\nabla_u\phi(u(t),w(t)), \tag{6}$$

satisfies $g(t)\to 0$ as $t\to\infty$. Note that $g(t)$ (as well as $w(t)$) is a quantity that cannot be evaluated by the controller (5), as $w(t)$ is unknown and unmeasurable.

[2]Although one could consider a more general control algorithm of the form $\dot z(t)=F_c(z(t),y(t))$ and $u(t)=G_c(z(t),y(t))$, we will show in Section IV that allowing for G_c to depend on $y(t)$ is unnecessary. Hence, we focus on the simpler formulation (5) for the sake of notation.

The dynamics of the plant (1), combined with the exosystem (4) and controller (5), form a nonlinear autonomous system:

$$\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), w(t)), & y(t) &= c(x(t), w(t)),\\
\dot{z}(t) &= F_c(z(t), y(t)), & u(t) &= G_c(z(t)),\\
\dot{w}(t) &= s(w(t)), & g(t) &= \nabla_u \phi(u(t), w(t)).
\end{aligned} \tag{7}$$

Since one of our objectives is asymptotic stabilization, we assume the existence of an equilibrium $(x_\circ^\star, z_\circ^\star, u_\circ^\star, y_\circ^\star)$ of the closed-loop system (7) with zero disturbance[3]:

$$\begin{aligned}
&0 = f(x_\circ^\star, u_\circ^\star, 0), \quad 0 = F_c(z_\circ^\star, y_\circ^\star), \quad 0 = \nabla_u \phi(u_\circ^\star, 0),\\
&y_\circ^\star = c(x_\circ^\star, 0), \qquad u_\circ^\star = G_c(z_\circ^\star),
\end{aligned} \tag{8}$$

and we assume that this point is locally unique.

One more requirement is in order. Because one of our objectives is output stabilization of (7), it is necessary to impose standard output stabilization assumptions. To state our conditions, we introduce the Jacobian matrices of the functions in (7). Specifically, let $A \in \mathbb{R}^{n\times n}$, $P \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $Q \in \mathbb{R}^{q\times p}$, $B \in \mathbb{R}^{n\times m}$, and $S \in \mathbb{R}^{p\times p}$ be:

$$\begin{aligned}
A &:= \left[\frac{\partial f}{\partial x}\right]_{(x,u,w)=(x_\circ^\star,u_\circ^\star,0)}, & P &:= \left[\frac{\partial f}{\partial w}\right]_{(x,u,w)=(x_\circ^\star,u_\circ^\star,0)},\\
C &:= \left[\frac{\partial c}{\partial x}\right]_{(x,w)=(x_\circ^\star,0)}, & Q &:= \left[\frac{\partial c}{\partial w}\right]_{(x,w)=(x_\circ^\star,0)},\\
B &:= \left[\frac{\partial f}{\partial u}\right]_{(x,u,w)=(x_\circ^\star,u_\circ^\star,0)}, & S &:= \left[\frac{\partial s}{\partial w}\right]_{w=0}.
\end{aligned} \tag{9}$$

Moreover, define:

$$A_L := \begin{bmatrix} A & P \\ 0 & S \end{bmatrix}, \qquad C_L := \begin{bmatrix} C & Q \end{bmatrix}. \tag{10}$$

Assumption 5 (***Stabilizability and detectability***)**.** The pair (A, B) is stabilizable and the pair (C_L, A_L) is detectable. □

Observe that this assumption is consistent with the existing literature. In particular, it appears explicitly in [4], [13] and implicitly in [2], [5]. In the latter works, the plant is assumed to be pre-stabilized (thus implicitly requiring stabilizability); moreover, $w(t)$ is assumed bounded, which, together with plant stability, ensures detectability.

We are now ready to formalize the notion of exact tracking.

Definition 2 (***Exact asymptotic tracking***)**.** The controlled plant (7) is said to *(exactly) asymptotically track a critical trajectory* of (2) when the following conditions hold:

(D2a) The equilibrium $(x_\circ^\star, z_\circ^\star)$ of the autonomous system:

$$\begin{aligned}
\dot{x}(t) &= f(x(t), G_c(z(t)), 0),\\
\dot{z}(t) &= F_c(z(t), c(x(t), 0)),
\end{aligned} \tag{11}$$

is locally exponentially stable.

(D2b) There exists a neighborhood $\Upsilon \subseteq X \times Z \times W$ of $(x_\circ^\star, z_\circ^\star, 0)$ such that, for each initial condition $(x(0), z(0), w(0)) \in \Upsilon$, the solution of (7) satisfies $\lim_{t\to\infty} g(t) = 0$. □

[3]Notice that this point can be obtained by first finding $x_\circ^\star$ and $u_\circ^\star$ such that $0 = f(x_\circ^\star, u_\circ^\star, 0)$ and $0 = \nabla_u \phi(u_\circ^\star, 0)$, and then constructing $y_\circ^\star$ and $z_\circ^\star$ along with the controller F_c and G_c to satisfy the remaining conditions.

We are now ready to reformulate rigorously Problem 1.

Problem 2 (***FO algorithm design as an output regulation problem***)**.** Design, when possible, $F_c(z, y)$, $G_c(z)$, and n_c so that (7) asymptotically tracks a critical trajectory of (2). □

Together, Definition 2 and Problem 2 frame the FO problem (2) as an output regulation problem, where the signal to be regulated is $g(t)$.

IV. Existence and Design of FO Algorithms

In this section, we present conditions for the solvability of Problem 2 and present the proposed algorithm.

A. Solvability of Problem 2

The following definition, inspired by [28], is instrumental.

Definition 3 (***Limit point and limit set***)**.**

1) A point $\bar{w} \in W$ is said to be a *limit point of* (4) *with respect to (w.r.t.) the initialization* $w_\circ \in W$ if there exists a non-decreasing sequence $\{k_i\}_{i\in\mathbb{N}_{\geq 0}}$, with $k_i \to \infty$ as $i \to \infty$, such that the trajectory of (4) with $w(0) = w_\circ$ satisfies $w(k_i) \to \bar{w}$ as $i \to \infty$.
2) Given $w_\circ \in W$, we denote by $\Omega(w_\circ)$ the set of all limit points (i.e., for all sequences) of (4) w.r.t. the initialization $w_\circ$.
3) Given $W_\circ \subseteq W$, $\Omega(W_\circ) := \bigcup_{w_\circ \in W_\circ} \Omega(w_\circ)$, is called the *limit set w.r.t. initializations in* $W_\circ$. □

Intuitively, $\Omega(W_\circ)$ denotes the set of all limit points (equilibria, limit cycles, etc.) that can be reached by (4) when initialized at points in $W_\circ$. By Assumption 3, $\Omega(W_\circ)$ is contained in some neighborhood of the origin of $\mathbb{R}^p$, whose radius depends on the initialization set $W_\circ$.

We have the following solvability result for Problem 2.

Theorem 1 (***Solvability of Problem 2***)**.** Let Assumptions 1–5 hold. Problem 2 is solvable if and only if there exist a neighborhood $W_\circ \subset W$ of the origin in $\mathbb{R}^p$ and C^2 mappings $\pi : W_\circ \to X$ and $\gamma : W_\circ \to \mathbb{R}^m$ such that

$$\frac{\partial \pi}{\partial w} s(w) = f(\pi(w), \gamma(w), w), \tag{12a}$$

$$0 = \nabla_u \phi(\gamma(w), w), \tag{12b}$$

hold at all limit points $w \in \Omega(W_\circ)$ of (4). □

The proof of this result is presented in [26]. Theorem 1 asserts that the solvability of Problem 2 hinges on the existence of two mappings, $x = \pi(w)$ and $u = \gamma(w)$. These mappings must (in a neighborhood of the limit points of (4)): (i) render the gradient of the cost identically zero (cf. (12b)) and (ii) algebraically relate the plant dynamics $f(x, u, w)$ to the exosystem dynamics $s(w)$ (cf. (12a)).

B. Controller design methodology

We now present the FO controller proposed in this work. The design, illustrated in Fig. 1, relies on a separation principle:

(C1) A dynamic observer estimates the unmeasured states $x(t)$ and $w(t)$, producing the estimates $\hat{x}(t)$ and $\hat{w}(t)$.

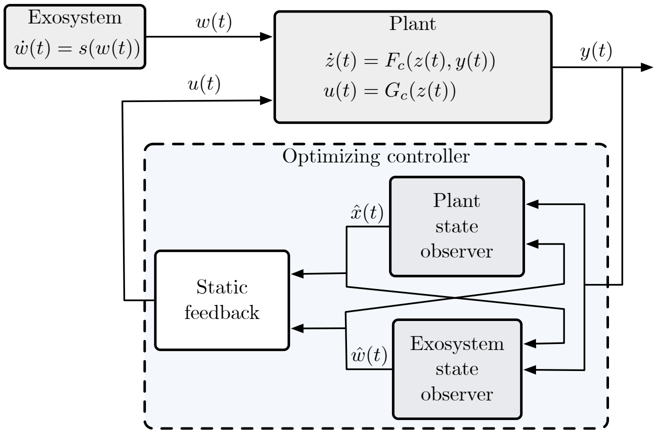

Fig. 1: Block-diagram representation of the optimizing controller developed in this work. The proposed controller is structured according to a separation principle, combining a state estimator (responsible for reconstructing the unmeasurable states $(\hat{x}(t), \hat{w}(t))$) with a static feedback law that regulates the plant toward a point where the gradient error vanishes.

(C2) A static feedback algorithm, based on the estimates $\hat{x}(t)$ and $\hat{w}(t)$, stabilizes the closed-loop (thus achieving (D2a)) and drives it to operating conditions where $g(t)$ vanishes (thus achieving (D2b)).

More precisely, the construction proceeds as follows. Given functions $\gamma(w)$ and $\pi(w)$ satisfying (12), we construct a controller with state $z = (z_1, z_2)$, where $z_1 \in \mathbb{R}^n$ and $z_2 \in \mathbb{R}^p$ (so that $n_c = n + p$), and dynamics

$$\begin{aligned}\dot{z}_1(t) &= f(z_1(t), u(t), z_2(t)) - L_1 e_y(t),\\ \dot{z}_2(t) &= s(z_2(t)) - L_2 e_y(t),\\ u(t) &= \gamma(z_2(t)) + K\big(z_1(t) - \pi(z_2(t))\big),\end{aligned} \tag{13}$$

where $e_y(t) = c(z_1(t), z_2(t)) - y(t)$. Intuitively, $z_1(t)$ operates as an estimator for $x(t)$, driven by the output error $e_y(t)$ with gain L_1. The state $z_2(t)$ operates as an estimator for $w(t)$, driven by the output error $e_y(t)$ with gain L_2. The input $u(t)$ combines a feedforward action (given by $\gamma(z_2(t))$), which is responsible for zeroing the gradient (cf. (12b)) with a plant stabilizing action (given by $K(z_1(t) - \pi(z_2(t)))$), both driven by the estimated states $z_1(t)$ and $z_2(t)$.

In (13), the matrices $K \in \mathbb{R}^{m \times n}, L_1 \in \mathbb{R}^{n \times q}$, and $L_2 \in \mathbb{R}^{p \times q}$ are designed such that the closed-loop matrices:

$$A + BK \qquad \text{and} \qquad A_L - LC_L, \tag{14}$$

where $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}$, have eigenvalues[4] in $\mathbb{C}_< := \{s : \operatorname{Re} s < 0\}$. Intuitively, Hurwitz stability of $A_L - LC_L$ guarantees asymptotic stability of the dynamic observer, while Hurwitz stability of $A + BK$ ensures that the state-feedback controller $u(t)$ renders the closed-loop system asymptotically stable.

We summarize our design procedure in Algorithm 1, and we formalize the effectiveness of this method in Theorem 2.

Theorem 2 (***Correctness of Algorithm 1***)**.** Let Assumptions 1–5 hold, and $F_c(z, y)$, $G_c(z)$ be designed according to Algorithm 1. Then the controlled system (7) asymptotically tracks a critical trajectory of (2). □

[4]Notice that existence of K and L is guaranteed by Assumption 5.

Algorithm 1: FO algorithm design

Data: Mappings $f(x, u, w)$, $c(x, w)$, $s(w)$; matrices A, B as in (9), A_L, C_L as in (10); mappings $\pi(w), \gamma(w)$ as in Theorem 1

Let K be any matrix such that $A + BK$ is Hurwitz;
Let L be any matrix such that $A_L - LC_L$ is Hurwitz;
Decompose $z = (z_1, z_2), z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^p$;
Decompose $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}, L_1 \in \mathbb{R}^{n \times q}, L_2 \in \mathbb{R}^{p \times q}$;
$G_c(z) \leftarrow \gamma(z_2) + K(z_1 - \pi(z_2))$;
$F_c(z, y) \leftarrow \begin{bmatrix} f(z_1, G_c(z), z_2) - L_1(c(z_1, z_2) - y) \\ s(z_2) - L_2(c(z_1, z_2) - y) \end{bmatrix}$;

Result: $F_c(z, y)$, $G_c(z)$ that solve Problem 2

We note that the proposed design requires the maps $\pi(w)$ and $\gamma(w)$, which may be available in closed form or computed using standard numerical approximation techniques. We refer to [26] for additional discussions on computing these maps.

V. Illustrative application of Algorithm 1

Consider the following instance of (1):

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Pw(t),\\ y(t) &= Cx(t) + Qw(t),\end{aligned}$$

where $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times m}, P \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{q \times n}, Q \in \mathbb{R}^{q \times p}$, and the following instance of (2) with $\lambda \geq 0$:

$$\begin{aligned}&\underset{u \in \mathbb{R}^m,\, x \in \mathbb{R}^n}{\text{minimize}} \quad \tfrac{1}{2}\|x\|^2 + \tfrac{\lambda}{2}\|u\|^2,\\ &\text{subject to:} \quad 0 = Ax + Bu + Pw(t).\end{aligned} \tag{15}$$

When A is invertible, Assumption 2 is satisfied with $x = h(u, w) = -A^{-1}Bu - A^{-1}Pw$, and the problem reduces to an unconstrained quadratic optimization in u. Restricting, for simplicity, to linear controllers, our search class (5) is:

$$\dot{z}(t) = A_c z(t) + B_c y(t), \qquad u(t) = C_c z(t). \tag{16}$$

Problem 2 thus consists of designing A_c, B_c, C_c to achieve exponential stability of the closed-loop system and asymptotic regulation of the gradient signal:

$$g(t) = Ru(t) + Tw(t),$$

where $R = B^\mathsf{T} A^{-\mathsf{T}} A^{-1} B + \lambda I$ and $T = B^\mathsf{T} A^{-\mathsf{T}} A^{-1} P$.

By Theorem 1, Problem 2 is solvable if and only if there exist $\Pi \in \mathbb{R}^{n \times p}$ and $\Gamma \in \mathbb{R}^{m \times p}$ such that

$$\Pi S = A\Pi + B\Gamma + P, \qquad 0 = R\Gamma + T. \tag{17}$$

For this problem, Algorithm 1 reads as:

1) Let K be any matrix such that $A + BK$ is Hurwitz;
2) Let L be any matrix such that $A_L - LC_L$ is Hurwitz;
3) Decompose $z = (z_1, z_2), z_1 \in \mathbb{R}^n, z_2 \in \mathbb{R}^p$;
4) Decompose $L = [L_1^\mathsf{T}, L_2^\mathsf{T}]^\mathsf{T}, L_1 \in \mathbb{R}^{n \times q}, L_2 \in \mathbb{R}^{p \times q}$;
5) Design the map $G_c(z)$ as follows:

$$G_c(z) = \Gamma z_2 + K(z_1 - \Pi z_2) = C_c z.$$

That is, $C_c = \begin{bmatrix} K & \Gamma - K\Pi \end{bmatrix}$;

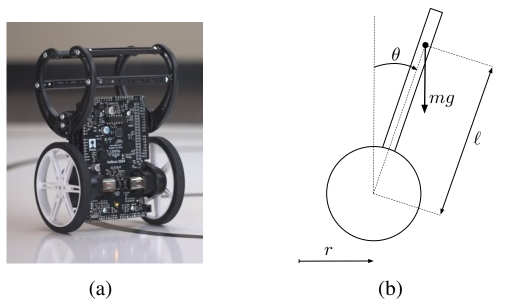

Fig. 2: (a) Illustration of the balancing robot discussed in Section VI. (b) Free-body diagram of the robot.

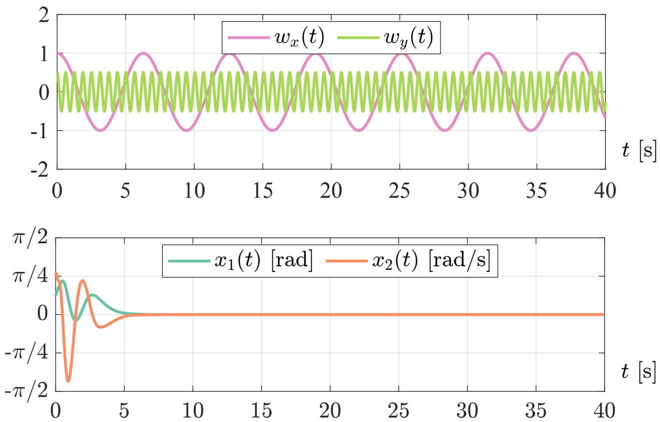

Fig. 3: Control performance when applied to the problem (22) for the balancing robot in Fig. 2(a) (see the model in (20)). Despite the presence of unmeasurable time-varying disturbances affecting both the states and the output (top panel), the controller successfully regulates the system states to the origin (bottom panel). See Fig. 4 for the corresponding gradient error.

6) Design the map $F_c(z, y)$ as follows:

$$F_c(z, y) = \begin{bmatrix} Az_1 + BC_c z + Pz_2 - L_1(Cz_1 + Qz_2 - y) \\ Sz_2 - L_2(Cz_1 + Qz_2 - y) \end{bmatrix}.$$

That is, $F_c(z, y) = A_c z + B_c y$, with

$$A_c = \begin{bmatrix} A + BK - L_1 C & P + B(\Gamma - K\Pi) - L_1 Q \\ -L_2 C & S - L_2 Q \end{bmatrix},$$

$$B_c = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix}. \qquad \square$$

VI. Application to Robotic Balancing Control

Consider a two-wheeled robot moving on a surface, as shown in Fig. 2(a), with the control objective of balancing it around the vertical position. Let θ denote the rotation angle of the center of mass and r the horizontal displacement of the wheels. The model in Fig. 2(b) gives:

$$J_e \ddot{\theta}(t) = mg\ell \sin\theta(t) - k\ell^2 \dot{\theta}(t) - m\ell \ddot{r}(t)\cos\theta(t) + w_x(t), \tag{18}$$

where $J_e > 0$ is the moment of inertia of the rod, $m > 0$ its mass, $\ell > 0$ its length, g is the gravitational acceleration, and $k > 0$ models a frictional force. The signal $w_x(t)$ models external disturbances, such as uneven surfaces.

Suppose the robot is equipped with an Inertial Measurement Unit (IMU) providing noisy measurements of $\dot{\theta}(t)$:

$$y(t) = \dot{\theta}(t) + w_y(t), \tag{19}$$

where $w_y(t)$ models sensor noise. Viewing the cart's acceleration $\ddot{r}(t)$ as the control (i.e., letting $u(t) := \ddot{r}(t)$), letting $x_1 = \theta$ and $x_2 = \dot{\theta}$, a state-space realization for (18) is:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \alpha \sin x_1(t) - \beta x_2(t) - \gamma u(t)\cos x_1(t) + \eta w_x(t), \\ y(t) &= x_2(t) + w_y(t), \end{aligned} \tag{20}$$

where $\alpha = \frac{mg\ell}{J_e}, \beta = \frac{k\ell^2}{J_e}, \gamma = \frac{m\ell}{J_e}$, and $\eta = \frac{1}{J_e}$. For our simulations, we used $\ell = 0.023$ [m], $m = 0.316$ [kg], $k = 0.1$ [kg/s], $g = 9.81$ [m/s^2], and $J_e = 0.000444$ [kg m^2].

We consider the following disturbance signals: $w_x(t) = \rho_x \cos(\bar{\omega}_x t)$, $\rho_x = 1$ [rad/s^2], $\bar{\omega}_x = 1$ [rad/s], $w_y(t) = \rho_y \cos(\bar{\omega}_y t)$, $\rho_y = 0.5$ [rad/s], and $\bar{\omega}_y = 10$ [rad/s]. These signals can be generated by an exosystem as in (4) with state $w = (w_1, w_2, w_3, w_4) \in \mathbb{R}^4$, $s(w) = Sw$, with

$$S = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\bar{\omega}_x^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\bar{\omega}_y^2 & 0 \end{bmatrix}, \tag{21}$$

and initialized at $w(0) = (\rho_x, 0, \rho_y, 0)$. In particular, $w_x(t)$ and $w_y(t)$ correspond to the first and third states of this model; i.e., $w_x(t) = w_1(t)$ and $w_y(t) = w_3(t)$. A mapping $h(u, w)$ satisfying Assumption 2 is given by:

$$h(u, w) = \begin{bmatrix} -2\tan^{-1}\left(\frac{\alpha - \sqrt{\alpha^2 - \eta^2 w_1^2 + \gamma^2 u^2}}{\eta w_1 + \gamma u}\right) \\ 0 \end{bmatrix},$$

when $\eta w_1 + \gamma u \neq 0$ and $h(u, w) = [\pi, 0]^\mathsf{T}$ otherwise. We consider the following instance of (2):

$$\begin{aligned} &\underset{u\in\mathbb{R},\, x\in\mathbb{R}^2}{\text{minimize}} && \tfrac{1}{2}\|x\|^2, \\ &\text{subject to:} && 0 = x_2, \\ & && 0 = \alpha \sin x_1(t) - \beta x_2(t) \\ & && \qquad - \gamma u(t)\cos x_1(t) + \eta w_x(t), \end{aligned} \tag{22}$$

which formalizes the objective of balancing the robot at the vertical position (where $\|x\|$ is minimized) while rejecting disturbances. To determine mappings $\pi(w)$ and $\gamma(w)$ that solve (12), we sought an approximate solution using polynomial representations:

$$\pi(w) = \sum_{\ell=1}^{d_\pi} \langle \psi_\ell^\pi, \Theta_\ell(w) \rangle, \quad \gamma(w) = \sum_{\ell=1}^{d_\gamma} \langle \psi_\ell^\gamma, \Theta_\ell(w) \rangle, \tag{23}$$

of order d_π and d_γ, respectively. Here, ψ_ℓ^π and ψ_ℓ^γ are $\binom{\ell p}{\ell}$-dimensional vectors of coefficients and $\Theta_\ell(w)$, $\ell \in \mathbb{N}_{>0}$, is an ℓ-th order polynomial basis of functions; formally:

$$\begin{aligned} \Theta_\ell(w) = [&w_1^\ell, w_1^{\ell-1} w_2, \dots, w_1^{\ell-1} w_p, \\ &w_1^{\ell-2} w_2^2, w_1^{\ell-2} w_2 w_3, \cdots, w_1^{\ell-2} w_2 w_p, \cdots, w_p^\ell]^\mathsf{T}. \end{aligned}$$

The parameter vectors ψ_ℓ^π and ψ_ℓ^γ have been fitted numerically so that (12) holds in a neighborhood of the limit points of $w(t)$. In our simulations, we used $d_\pi = d_\gamma = 4$, and (12) was satisfied up to a relative error of 10^{-7}. Note that,

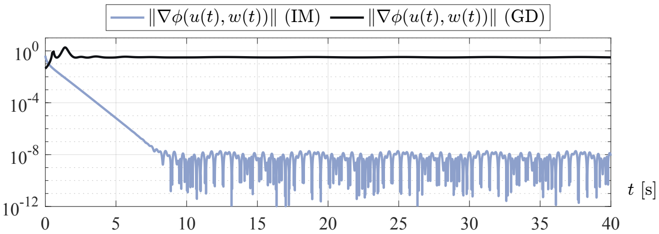

Fig. 4: *Light blue line*: Gradient error for the proposed internal-model-based controller (IM) applied to the scenario in Fig. 3. *Black line*: Gradient error obtained with the gradient-based controller (GD) from [2]. The results show that the proposed controller drives the gradient error to zero up to numerical precision, whereas the feedback-optimization approach does not converge due to the absence of internal-model dynamics.

to solve the internal-model conditions (12), knowledge of the magnitude of the disturbance signals ρ_x and ρ_y is not needed; it is only required to know their frequencies $\bar{\omega}_x$ and $\bar{\omega}_y$ (which are the only parameters in S). We applied Algorithm 1, selecting the controller gain K and the observer gain L so that the eigenvalues of $A+BK$ and A_L-LC_L are uniformly distributed in $[-1,-2]$ and $[-2,-3]$, respectively. Simulation results are illustrated in Fig. 3. The plots show that the control methodology is effective at regulating the pendulum to the vertical position $(x_1,x_2)=(0,0)$. Fig. 4 illustrates that the methodology is successful at regulating the gradient error signal $\|g(t)\|$ to zero, up to a numerical error of order 10^{-6}. We attribute this to numerical errors related to fitting (23). The plot also compares our method with the algorithm from [2], [7], which fails to find the exact optimizer since it is effective only when $w(t)$ is constant.

VII. Conclusions

We have shown that FO problems can be recast as output regulation problems. This allowed us to propose a novel FO algorithm, inspired by output regulation, applicable to plants subject to non-constant disturbances. The method combines an output-feedback control action stabilizing the plant with a control action that drives the system toward the set of critical points. This work opens the opportunity for several directions, including an investigation of methods that use inexact internal model knowledge or learn the internal model online, the relaxation of the convexity–smoothness assumptions, and an investigation of discrete-time algorithms. Extensions to algorithms that attain global convergence are also an important research direction.

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