

Temporal Variabilities Limit Convergence Rates in Gradient-Based Online Optimization

Bryan Van Scoy

Gianluca Bianchin

Abstract— This paper investigates the fundamental performance limits of gradient-based algorithms for time-varying optimization. Leveraging the internal model principle and root locus techniques, we show that temporal variabilities impose intrinsic limits on the achievable rate of convergence. For a problem with condition ratio κ and time variation whose model has degree n , we show that the worst-case convergence rate of any minimal-order gradient-based algorithm is $\rho_{TV} = \left(\frac{\kappa-1}{\kappa+1}\right)^{1/n}$. This bound reveals a fundamental tradeoff between problem conditioning, temporal complexity, and rate of convergence. We further construct explicit controllers that attain the bound for low-degree models of time variation.

I. INTRODUCTION

Time-varying optimization problems provide a natural framework to describe decision-making tasks in which objectives and/or constraints evolve dynamically over time. Such problems arise in diverse engineering domains, including online learning and streaming data in machine learning [1], [2], adaptive filtering in signal processing [3], and trajectory planning or model predictive control in robotics [4].

Historically, the study of time-varying optimization has relied on continuity arguments with respect to static formulations: when the temporal variations of the problem are sufficiently slow, algorithms developed for static problems produce nearly optimal solutions in the time-varying setting [5]. This reasoning suggests that algorithms for time-varying problems may exhibit the same fundamental performance characteristics as those for time-invariant problems. For example, it is well established that the best achievable rate of gradient descent on time-invariant problems is $\rho_{TI} = \frac{\kappa-1}{\kappa+1}$, where κ denotes the condition ratio of the problem [6]. Accordingly, one may be tempted to conclude that algorithms for time-varying problems can attain the same rate, provided that the temporal variability is ‘slow enough.’ In contrast, we prove a fundamental relationship between the convergence rate ρ , the condition ratio κ , and the number n of modes that characterize the temporal variability of the problem. Since there is no direct relation between n and ‘how fast’ the problem varies, but rather with the *complexity* of its temporal structure, our results reveal fundamental differences between static optimization problems and their dynamic counterparts.

This material is based upon work supported in part by the National Science Foundation under Award No. 2347121 and in part by the FRFS WEL-T Investigator Programme. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

B. Van Scoy is with the Dept. of Electrical and Computer Engineering, Miami University, OH 45056, USA. Email: bvanscoy@miamioh.edu

G. Bianchin is with the ICTEAM institute and the Department of Mathematical Engineering (INMA) at the University of Louvain, Belgium. Email: gianluca.bianchin@uclouvain.be

Contributions. Our main contributions are as follows:

- 1) We provide a fundamental bound on the worst-case convergence rate of minimal-degree controllers for unconstrained quadratic time-varying optimization, which is

$$\rho_{TV} := \left(\frac{\kappa-1}{\kappa+1}\right)^{1/n}, \quad (1)$$

where κ is the condition ratio of the objective, and n is the number of modes in the model of the time variation. Since quadratics are a special case of convex, Lipschitz-smooth objectives, this lower bound also applies as fundamental limitations to this broader class.

- 2) We use root locus techniques to design explicit controllers for particular models of the time variation.

Related works. The literature on time-varying optimization methods can be broadly divided into two classes. The first class comprises approaches that ignore or do not exploit any model of the temporal variability, and instead solve a sequence of static problems [7]–[9]. These methods only react after changes are observed, and therefore incur a certain regret and achieve at best convergence to a neighborhood of the optimizer [5]. The second class, instead, leverages a model of the temporal evolution of the problem to track the optimal trajectory exactly. Indeed, such a model is necessary for exact tracking [10], [11]. A prominent example is the prediction-correction framework [12], [13], where each step combines a prediction of the optimizer’s evolution with a correction based on the current problem. Extensions of these ideas have recently been studied under contraction analysis [4], sampling-based estimation of variability [14], and constrained formulations [15]; see also the survey [16].

There has recently been a spark of interest in using control tools to design optimization algorithms. Fundamental results have been derived in [10] for discrete-time problems and [11] for continuous-time ones. For problems with quadratic objectives, root locus and the internal model principle can be used to analyze and design the optimization method [17]. Recently, [18] have investigated constrained settings and stochastic problems. Particularly relevant to this work are [19], [20], which considered quadratic objectives with polynomial temporal variabilities. In [20], the authors use Nevanlinna–Pick interpolation to establish a fundamental lower bound of $\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^{1/n}$ (cf. (1)). These results, however, are restricted to polynomial temporal variabilities and focus on accelerated methods, which are known to potentially fail to achieve global convergence on more general objectives (beyond quadratics) [6], and are also susceptible to noise

amplification in gradient evaluations [21]. For these reasons, we restrict our attention to the class of non-accelerated (minimal-order) methods and general temporal variabilities.

Notation: We denote by \mathbb{N} and \mathbb{R} the sets of natural and real numbers, respectively; by $\mathbb{R}[z]$ the space of real-coefficient polynomials in z ; and by $\mathbb{R}^d[z]$ the space of d -dimensional vector polynomials in z with real coefficients.

II. PROBLEM SETUP

We consider time-varying unconstrained optimization problems, consisting of minimizing the quadratic objective

$$f_k(x) = \frac{1}{2}x^\top A x + b_k^\top x \quad (2)$$

with time indexed by $k \in \mathbb{N}$. Here, $x \in \mathbb{R}^d$ is the decision variable, $A \in \mathbb{R}^{d \times d}$ is a symmetric time-invariant matrix, and $b = \{b_k\}_{k \in \mathbb{N}}$ with $b_k \in \mathbb{R}^d$ is a time-varying parameter.

Remark 1 (Quadratic objective functions): We focus on the class of quadratic objectives as they provide a structured framework to derive bounds on the worst-case convergence rate. Since quadratics are a special case of convex, Lipschitz-smooth objectives, and our goal is to establish lower bounds on the convergence rate, the forthcoming estimates also serve as fundamental limitations for this broader function class. This is in line with the time-invariant case, where [22, Sec. 2.1.4] shows that the optimal convergence rate achievable by any iterative algorithm on problems with convex and Lipschitz-smooth loss is attained for quadratics. \square

We make the following assumptions throughout.

Assumption 1 (Eigenvalues of A): The matrix A has eigenvalues in the closed interval $[\mu, L]$ with $0 < \mu < L$. Moreover, the parameters μ and L are known. \square

By Assumption 1, the cost (2) is strongly convex with parameter μ , and the gradient is Lipschitz smooth with parameter L . Considering problems of this class is a standard assumptions in optimization [9], which has been widely used in related works [23]–[25]. In what follows, we let $\kappa := L/\mu$ denote the *condition ratio* of the objective in (2).

We make the following assumption on $B(z)$, the \mathcal{Z} -transform of the time-varying sequence $b = \{b_k\}_{k \in \mathbb{N}}$.

Assumption 2 (Model of time variation): $B(z)$ is a rational function of z with all poles in $\mathbb{D}_{=1} := \{z : |z| = 1\}$. \square

Assumption 2 specifies the class of temporal variations of the parameter b under consideration. Intuitively, signals whose \mathcal{Z} -transform is rational in z are signals generated by finite-order linear recurrences (i.e., by LTI systems with a finite number of states) with poles that lie on the unit circle. This class includes a wide variety of signals including causal periodic sequences (sinusoids, square waves, etc.) and polynomial sequences (constant, ramp, parabolic, etc.). Similar assumptions have been employed in related works [17], [20].

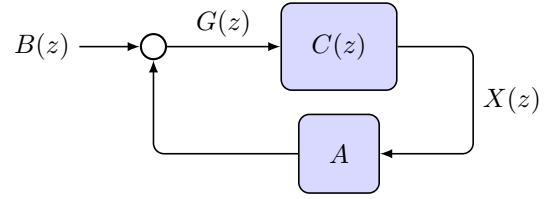


Fig. 1. Structure of the gradient-based optimization algorithms, as a block-diagram in the frequency domain. See (5).

By Assumption 2, $B(z)$ admits the representation

$$B(z) = \frac{B_N(z)}{m(z)}, \quad (3)$$

where $B_N(z) \in \mathbb{R}^d[z]$ and $m(z) \in \mathbb{R}[z]$. Without loss of generality, we use the notation

$$m(z) = z^n + \sum_{i=0}^{n-1} m_i z^i, \quad m_i \in \mathbb{R},$$

where n is the number of poles¹ of $B(z)$.

In line with the literature [22], we focus on gradient-type algorithms for minimizing (2); that is, algorithms that have access to oracle evaluations of the gradient of (2):

$$(k, x) \mapsto \nabla f_k(x) = Ax + b_k. \quad (4)$$

We restrict our attention to optimization algorithms whose iterates are obtained by processing (4) through Linear Time-Invariant (LTI) filters. Precisely, let $x_k \in \mathbb{R}^d$ denote the estimate for the minimizer of (2) generated by the algorithm² at time $k \in \mathbb{N}$, and by $X(z)$ its \mathcal{Z} -transform. Then, we consider algorithms that generate x_k as:

$$X(z) = C(z)G(z) = C(z)(AX(z) + B(z)), \quad (5)$$

where $C(z) \in \mathbb{R}^{d \times d}[z]$ is the transfer function of the filter, and $G(z) = AX(z) + B(z)$ is the \mathcal{Z} -transform of the gradient $\nabla f_k(x_k)$. This algorithm structure is illustrated in Fig. 1. We make the following assumption on the filter throughout, which specifies that $C(z)$ is a linear filter, with the additional requirement that it be strictly proper to guarantee real-time implementability of the algorithm.

Assumption 3 (Structure of the optimization filter): $C(z)$ is a rational, strictly proper function of z . \square

In what follows, we focus on designing optimization filters with optimal rate of convergence and that are of minimal order; we make these two notions formal next.

¹Note that (3) does not restrict the poles of $B(z)$ to be identical for each component, since $m(z)$ can be chosen as the polynomial whose root set is the union of the root sets of the individual components of $B(z)$.

²Since our focus is on characterizing the asymptotic convergence rate of the method, we henceforth assume that the internal state of the optimization filter is initialized to zero, noting that the framework can be extended to nonzero initial conditions by accounting for the free response of $C(z)$, an extension we leave for future work due to space limitations.

Definition 1 (Asymptotic tracking): We say that (5) asymptotically tracks the minimizer of (2) if the sequence $\{x_k\}$ satisfies

$$\lim_{k \rightarrow \infty} \|x_k - x_k^*\| = 0,$$

where $x_k^* := -A^{-1}b_k$ is the minimizer of (2). Moreover, the root-convergence factor (or, simply, convergence rate) is

$$\rho := \limsup_{k \rightarrow \infty} \|x_k - x_k^*\|^{1/k}. \quad \square$$

Note that Definition 1 formalizes a notion of *exact* tracking, whereby x_k reaches x_k^* with *zero error*, asymptotically.

Remark 2: The minimizer x_k^* of the quadratic objective (2) has \mathcal{Z} -transform $X^*(z) = -A^{-1}B(z) = -\frac{1}{m(z)}A^{-1}B_N(z)$. Consequently, $B(z)$ and $X^*(z)$ share the same poles. It follows that Assumption 2 can equivalently be formulated in terms of a model for x_k^* rather than b_k , aligning our formulation with other models in the literature (e.g., [20]). \square

Definition 2 (Optimization filters of minimal order): Let $C(z)$ be an optimization filter that asymptotically tracks the minimizer of (2). We say that $C(z)$ is of *minimal order* if the degree of its denominator polynomial is minimal among all filters that asymptotically track the minimizer. \square

Although higher-order filters could be employed, yielding accelerated algorithms [26], our focus here is on the class of *non-accelerated* gradient methods [22]. Motivations for studying non-accelerated methods include that accelerated methods are known to possibly fail to achieve global convergence on more general objectives (beyond quadratics) [6] and also amplify noise in gradient evaluations [21].

We are now ready to formalize the objective of this work.

Problem 1: Determine the optimal worst-case convergence rate achievable by any minimal-order optimization algorithm of the form (5), where *optimal* is with respect to all optimization filters of minimal order satisfying Assumption 3, and *worst-case* is with respect to all objectives of the form (2) satisfying Assumptions 1 and 2. In addition, construct an optimization filter that attains this rate. \square

III. PRELIMINARIES

In this section, we provide an instrumental characterization of the convergence rate that will enable us to address Problem 1. We begin with the following result.

Lemma 1 (Fundamental structure of tracking filters): Let Assumptions 1 to 3 hold, and consider optimization filters of the form

$$C(z) = \frac{C_N(z)}{m(z)}, \quad C_N(z) \in \mathbb{R}^{d \times d}[z]. \quad (6)$$

(L1) Suppose $C(z)$ is an optimization filter that achieves exact asymptotic tracking. Then, $C(z)$ is of minimal order only if it has the form (6).

(L2) Suppose $C(z)$ has the form (6) and the roots of $\det(m(z)I - AC_N(z))$ are in $\mathbb{D}_{<1} := \{z : |z| < 1\}$.

Then, $C(z)$ is an optimization filter of minimal order that achieves exact asymptotic tracking.

Proof: Solving (5) for the gradient yields

$$G(z) = (I - AC(z))^{-1}B(z).$$

By Assumption 3, we can write $C(z) = \frac{C_N(z)}{c_D(z)}$ with $C_N(z) \in \mathbb{R}^{d \times d}[z]$ and $c_D(z) \in \mathbb{R}[z]$. Substituting this form and (3) into the expression for the gradient yields

$$G(z) = \frac{c_D(z)}{m(z)} (c_D(z)I - AC_N(z))^{-1} B_N(z). \quad (7)$$

The filter achieves exact asymptotic tracking if and only if all poles of $G(z)$ are strictly inside the unit circle. Each entry of $(c_D(z)I - AC_N(z))^{-1}$ is a rational function of z , and the denominator polynomial is $\det(c_D(z)I - AC_N(z))$. Since all roots of $m(z)$ are marginally stable by Assumption 2, the poles of $G(z)$ would lie on the unit circle unless the poles introduced by $m(z)$ were canceled by either $c_D(z)$, $B_N(z)$, or the adjugate of $c_D(z)I - AC_N(z)$. Since the roots of $m(z)$ are poles of $B(z)$, they are not canceled by each component of $B_N(z)$. Moreover, they cannot be canceled by the adjugate for all A satisfying Assumption 1. Thus, for $C(z)$ to achieve exact asymptotic tracking with minimal order, its denominator must take the form $c_D(z) = m(z)$ as claimed. Finally, the choice $C(z)$ in (6) includes precisely this necessary factor and no additional pole factors, and hence is of minimal order. \blacksquare

Lemma 1 provides necessary and sufficient conditions for asymptotic tracking. The statement (L1) provides a necessary condition for an optimization filter to be of minimal order; the property that $C(z)$ is required to incorporate precisely the same poles as $B(z)$ can be interpreted as an instance of the *internal model principle of time-varying optimization* [10], [11], as it captures the requirement that the optimization filter must embed an internal model of the temporal variability of the problem (encoded by $m(z)$). Conversely, the statement (L2) provides a sufficient condition for an optimization filter to be of minimal order and achieve exact tracking, requiring that all roots of $\det(m(z)I - AC_N(z))$ to be in the open unit disk. This condition will be used later in this work to construct algorithms that address Problem 1.

Because we search within the class of optimization filters of minimal order, driven by the conclusion of (L2), we assume the optimization filter has the form (6). Moreover, we also assume that the same filter is applied to each component of the objective function.

Assumption 4 (Optimization filter is of minimal order):

The optimization filter has the form $C(z) = c(z)I_d$, where $c(z) = \frac{d(z)}{m(z)}$ for some $d(z) \in \mathbb{R}[z]$. \square

We next derive an equivalent reformulation of the convergence rate (cf. Definition 1) for this class of algorithms that will be instrumental to derive our main results.

Lemma 2 (Characterization of the convergence rate):

Suppose Assumptions 1 to 4 hold. The optimal worst-

case convergence rate achievable by any minimal-order optimization algorithm of the form (5) is given by the optimal value of the following min-max problem:

$$\begin{aligned} \rho &= \min_{d(z) \in \mathbb{R}[z]} \max_{\lambda \in [\mu, L]} |z| \\ &\text{subject to } m(z) - \lambda d(z) = 0. \end{aligned} \quad (8)$$

Proof: Let $A = V\Lambda V^\top$ be an eigendecomposition of A , with $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_d)$ and V an orthonormal matrix of (real) eigenvectors. Let $\tilde{x}_k := V^\top x_k$ and $\tilde{X}(z)$ denote the corresponding \mathcal{Z} -transform. By projecting (5) onto the range space of V , we obtain the update:

$$\tilde{X}(z) = \frac{d(z)}{m(z)} (\Lambda \tilde{X}(z) + V^\top B(z)).$$

With this decomposition, the iterates of (5) separate into d decoupled equations index by $i = 1, \dots, n$, given by

$$\tilde{X}_i(z) = \frac{d(z)}{m(z)} (\lambda_i \tilde{X}_i(z) + v_i^\top B(z)).$$

Rewriting this equation in input-output form gives

$$\tilde{X}_i(z) = \frac{d(z)}{m(z) - \lambda_i d(z)} v_i^\top B(z). \quad (9)$$

Therefore, the closed-loop poles of (5) are the roots of $m(z) - \lambda_i d(z)$. Since the convergence rate of an LTI system is the maximum modulus of the poles of its transfer function, this gives the formulation (8). ■

Lemma 2 provides a mathematical reformulation of the optimal worst-case convergence rate achievable by any minimal-order optimization algorithm. The outer minimization over the polynomial $d(z)$ captures a search over all optimization filters of minimal order that satisfy Assumption 3, while the inner maximization over $\lambda \in [\mu, L]$ accounts for the worst-case scenario over the entire class of objective functions consistent with Assumption 1.

IV. BOUND ON THE WORST-CASE CONVERGENCE RATE

We now provide a bound on the worst-case convergence rate achievable by any minimal-order algorithm as in Fig. 1 that asymptotically tracks the minimizer of the quadratic objective (2). Our bound depends only on the condition ratio κ of the objective function and the degree n of the model of the time variation. Our main result is the following.

Theorem 1 (Bound on worst-case convergence rate): Let Assumptions 1 to 4 hold. The worst-case convergence rate achievable by any minimal-order optimization algorithm is bounded below by ρ_{TV} defined in (1), where $\kappa := L/\mu$ is the condition ratio and n is the degree of the model $m(z)$.

Proof: Suppose Assumptions 1 to 4 hold, and suppose the controller $c(z)$ is such that the algorithm in Fig. 1 asymptotically tracks a critical trajectory with rate $\rho \in (0, 1)$ for all quadratic objectives (2) satisfying these assumptions. Since the controller is minimal degree (by Assumption 4), it must have the form $c(z) = d(z)/m(z)$, where $d(z)$ and $m(z)$ are polynomials, the degree of the numerator $d(z)$ is strictly

less than that of the denominator $m(z)$, and the model $m(z)$ is monic of degree n with all roots on the unit circle. Define the characteristic polynomial of the subsystem in (9) for a general eigenvalue $\lambda \in [\mu, L]$ as

$$p_\lambda(z) := m(z) - \lambda d(z).$$

Since the degree of $d(z)$ is strictly less than that of $m(z)$ and the model is monic, $p_\lambda(z)$ is also monic of degree n . Our goal is then to construct a lower bound on the convergence rate ρ such that all roots of the polynomial $p_\lambda(z)$ are in the disk $\mathbb{D}_{\leq \rho} := \{z : |z| \leq \rho\}$ for all $\lambda \in [\mu, L]$.

Write the numerator of the optimization filter in terms of its coefficients as $d(z) = d_{n-1}z^{n-1} + \dots + d_0$, where each of the d_i coefficients may be zero. Also, denote the roots of the characteristic polynomial as $\zeta_1, \dots, \zeta_n \in \mathbb{C}$, which all have modulus at most ρ by assumption. In terms of its roots and the coefficients of the controller polynomials, the characteristic polynomial is then

$$p_\lambda(z) = z^n + \sum_{i=0}^{n-1} (m_i - \lambda d_i) z^i = \prod_{i=1}^n (z - \zeta_i).$$

Define the corresponding elementary symmetric polynomials

$$e_k(p_\lambda) = \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \zeta_{i_1} \zeta_{i_2} \dots \zeta_{i_k}$$

for $k = 1, \dots, n$, which is the sum of all distinct products of k roots. Vieta's formulas relate the characteristic polynomial coefficients to the elementary symmetric polynomials by

$$e_k(p_\lambda) = (-1)^k (m_{n-k} - \lambda d_{n-k}).$$

Since $e_k(p_\lambda)$ is a sum of products of k roots, each root has modulus at most ρ , and there are $c_k := \binom{n}{k}$ terms in the summation, the elementary symmetric polynomials satisfy the bound

$$|e_k(p_\lambda)| \leq c_k \rho^k, \quad \forall \lambda \in [\mu, L] \text{ and } k = 1, \dots, n. \quad (10)$$

We can then bound the size of the numerator polynomial coefficients by

$$\begin{aligned} (L - \mu) |d_{n-k}| &= |(m_{n-k} - \mu d_{n-k}) - (m_{n-k} - L d_{n-k})|, \\ &\leq |m_{n-k} - \mu d_{n-k}| + |m_{n-k} - L d_{n-k}|, \\ &\leq 2c_k \rho^k, \end{aligned}$$

where the first inequality follows from the triangle inequality and the second inequality follows from the bound (10) applied to both endpoints $\lambda = \mu$ and $\lambda = L$. Again using the bound (10) at the endpoint $\lambda = \mu$ along with the (reverse) triangle inequality, we bound the size of the coefficients by

$$|m_{n-k}| - \mu |d_{n-k}| \leq |m_{n-k} - \mu d_{n-k}| \leq c_k \rho^k.$$

Combining this with the bound on $|d_{n-k}|$ above yields

$$\begin{aligned} |m_{n-k}| &\leq c_k \rho^k + \mu |d_{n-k}|, \\ &\leq c_k \rho^k + \mu \frac{2c_k}{L - \mu} \rho^k, \\ &= c_k \rho^k \frac{\kappa + 1}{\kappa - 1}, \end{aligned}$$

where $\kappa := L/\mu$ is the condition ratio. Isolating the convergence rate and using that this holds for all $k = 1, \dots, n$ yields the lower bound

$$\rho \geq \max_{k=1, \dots, n} \left(\frac{|m_{n-k}|}{c_k} \cdot \frac{\kappa - 1}{\kappa + 1} \right)^{1/k}. \quad (11)$$

Again using Vieta's formulas, each model coefficient m_{n-k} is (up to a sign) the sum of all distinct products of k roots, each of which has unit modulus by Assumption 2. Therefore, $|m_{n-k}| = e_k(m) \leq c_k$ for all $k = 1, \dots, n$. Moreover, this holds with equality when $k = n$, which produces the lower bound in (1). ■

Theorem 1 provides a fundamental lower bound on the rate of convergence attainable by any minimal-order optimization filter. The bound is expressed solely in terms of the properties of the optimization (the condition ratio κ of the objective) and the degree n of the temporal variability model $m(z)$. The dependence on κ is classical and in line with time-invariant counterparts: as the problem becomes more ill-conditioned (i.e., κ grows), the lower bound approaches one, indicating arbitrarily slow convergence in the worst case. The role of n quantifies a new, fundamental bound intrinsic to time-varying problems: as the temporal variability to be tracked becomes more complex (i.e., as n increases), the algorithm's convergence rate necessarily degrades, highlighting that *high-order internal models inherently preclude fast convergence*. Thus, (1) characterizes an intrinsic tradeoff between problem conditioning and model complexity, highlighting that even in the best-case design, the convergence rate cannot be improved beyond this limit. To compare, for the special case of polynomial models of the form $m(z) = (z - 1)^n$, the lower bound for *non-minimal* controllers is [20]

$$\rho \geq \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^{1/n}.$$

V. CONTROLLER DESIGN VIA ROOT LOCUS

We now shift to designing optimization filters of minimal order (see Definition 2) that satisfy Assumption 3 and attain the fundamental rate bound (1). Our approach is based on root locus techniques, which enables us to construct closed-form solutions to the optimization (8). Due to the complexity of this task, we focus on three specific cases: $n = 1, 2, 3$.

A. Case: $n = 1$

When the model has only a single pole, it must be either ± 1 since the poles are on the unit circle and the model coefficients are real (so complex roots must appear in conjugate pairs). We now consider these two cases.

First, suppose the model is $m(z) = z - 1$. Then we can use the standard gradient descent controller

$$c(z) = \frac{-\alpha}{z - 1} \quad \text{and} \quad \alpha = \frac{2}{L + \mu},$$

which achieves the optimal worst-case rate $\rho = \frac{\kappa - 1}{\kappa + 1}$. In this case, the root locus of $1 - \lambda c(z)$ starts at the open-loop pole

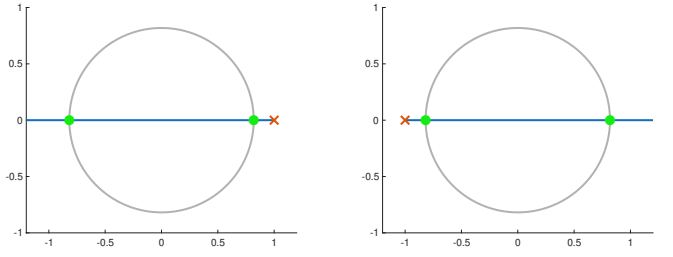


Fig. 2. Root locus with controller $-\alpha/(z-1)$ (left) and $\alpha/(z+1)$ (right). The locus (blue) starts at the open-loop poles (\times). The pole locations at gains $\lambda = \mu$ and $\lambda = L$ are shown (\bullet). For all $\lambda \in [\mu, L]$, the root locus is entirely contained in the ρ circle (gray).

of $z = 1$ when $\lambda = 0$ and moves to the left on the real axis as λ increases, crossing $z = \rho$ when $\lambda = \mu$ and $z = -\rho$ when $\lambda = L$. If the model is $m(z) = z + 1$, then the controller $c(z) = \frac{\alpha}{z+1}$ with the same stepsize α achieves the same rate. The root locus for both cases is shown in Fig. 2.

B. Case: $n = 2$

Now suppose the model has a single pair of complex conjugate poles on the unit circle with angle θ so that $n = 2$. We can then parameterize the controller as

$$c(z) = \frac{c_1 z - c_2}{z^2 - 2 \cos(\theta) z + 1}$$

with $c_1, c_2 \in \mathbb{R}$. The parameters that yield the optimal worst-case rate in (1) are

$$c_1 = \frac{-2 \cos \theta}{L} \quad \text{and} \quad c_2 = \frac{-2}{L + \mu},$$

which are the unique solutions to the condition that the root locus pass through both $z = -\rho$ and $z = \rho$ when $\lambda = L$. The root locus of this controller is shown in Fig. 3.

The previous case assumes the roots are complex conjugates. If instead the model has real roots at $z = +1$ and $z = -1$ so that $m(z) = z^2 - 1$, the optimal worst-case rate is achieved by the controller parameters $c_1 = 0$ and $c_2 = \frac{2}{L + \mu}$, which can be derived from the condition that the root locus pass through both $z = -\rho$ and $z = \rho$ when $\lambda = \mu$.

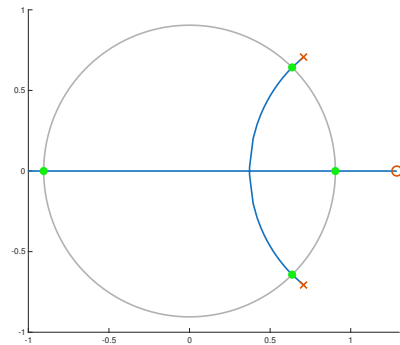


Fig. 3. Root locus with controller for a single frequency $\theta = \pi/4$. The locus (blue) starts at the open-loop poles (\times) and ends at the open-loop zeros (\circ). The pole locations at gains $\lambda = \mu$ and $\lambda = L$ are shown (\bullet). For all $\lambda \in [\mu, L]$, the root locus is entirely contained in the ρ circle (gray).

C. Case: $n = 3$

Now suppose the model has both a pair of complex conjugate poles and a pole at one,

$$m(z) = (z - 1)(z^2 - 2\cos(\theta)z + 1). \quad (12)$$

We can then parameterize the controller as

$$c(z) = \frac{c_2 z^2 + c_1 z + c_0}{m(z)}.$$

The optimal worst-case rate is achieved by the parameters

$$\begin{aligned} c_0 &= \frac{-1+\rho^3}{\mu}, \\ c_1 &= \frac{(-L+\mu+(L+\mu)\rho)(1+\rho^2)+2\rho(L+\mu-\rho(L-\mu))\cos(\theta)}{2\mu L\rho}, \\ c_2 &= -\frac{(-L+\mu+(L+\mu)\rho^2)(1+\rho)+2\rho(-L+\mu+(L+\mu)\rho)\cos(\theta)}{2\mu L\rho^2}, \end{aligned}$$

which yields the root locus in Fig. 4.

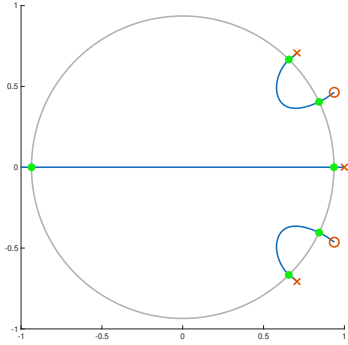


Fig. 4. Root locus with controller for both a constant and a single frequency $\theta = \pi/4$. The locus (blue) starts at the open-loop poles (\times) and ends at the open-loop zeros (\circ). The pole locations at gains $\lambda = \mu$ and $\lambda = L$ are shown (\bullet). For all $\lambda \in [\mu, L]$, the root locus is entirely contained in the ρ circle (gray).

VI. NUMERICAL SIMULATION

To illustrate our results, we simulate the trajectory of (5) using the optimization filter $C(z)$ designed in Section V to the quadratic objective in (2). We chose parameters $\mu = 1$, $L = 10$, and $d = 10$. The matrix A was randomly generated with eigenvalues uniformly distributed in $[\mu, L]$, and the time-varying linear term b_k was generated by an LTI system corresponding to the model $m(z)$ in (12) with $\theta = 0.01\pi$. The system trajectories are shown in Fig. 5. Note that the controller from [17] does not exist in this case, as the linear matrix inequalities to design the controller were infeasible.

VII. CONCLUSIONS

We have established fundamental limits on the attainable convergence rates of gradient-based algorithms for time-varying quadratic optimization. By leveraging tools from control theory, in particular the internal model principle and root locus techniques, we have shown that the optimal worst-case convergence rate necessarily degrades with the complexity of the temporal variability, quantified by the degree n of the underlying model. Overall, the results of this paper contribute to a deeper understanding of the intrinsic performance limits of gradient-based methods in

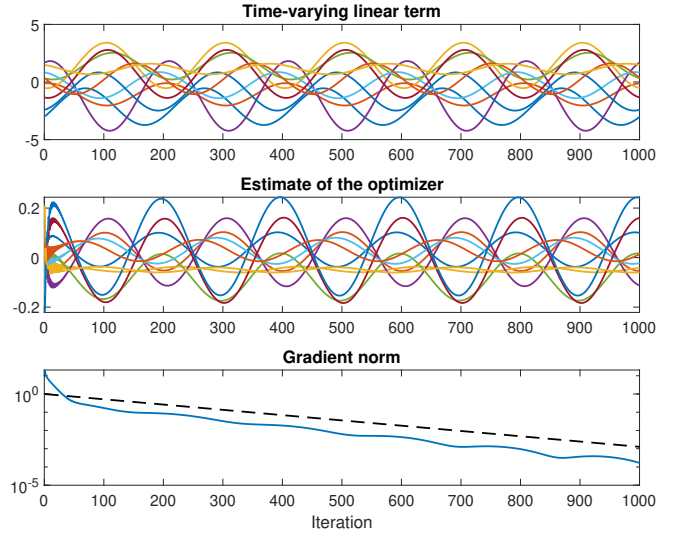


Fig. 5. Numerical simulation of the optimal worst-case controller from Section V-C to minimize the quadratic objective in (2). (Top) Trajectories of each component in the time-varying linear term b_k . (Middle) Trajectories of each component in the time-varying optimizer x_k^* . (Bottom) Gradient norm $\|Ax_k + b_k\|$ and the bound on the worst-case rate ρ^k (dashed). See Section VI for details.

time-varying optimization, showing that temporal variability fundamentally constrains the achievable rate of convergence. Avenues for future research are the incorporation of additional dynamics into the controller (beyond those strictly required by the internal model) to achieve *acceleration*, and considering more general function classes beyond quadratics.

REFERENCES

- [1] Y. Chen and Y. Zhou, “Machine learning based decision making for time varying systems: Parameter estimation and performance optimization,” *Knowledge-Based Systems*, vol. 190, p. 105479, 2020.
- [2] A. Rakhlin and K. Sridharan, “Online learning with predictable sequences,” in *Conference on Learning Theory*, 2013, pp. 993–1019.
- [3] F. Y. Jakubiec and A. Ribeiro, “D-map: Distributed maximum a posteriori probability estimation of dynamic systems,” *IEEE Transactions on Signal Processing*, vol. 61, no. 2, pp. 450–466, 2012.
- [4] A. Davydov, V. Centorrino, A. Gokhale, G. Russo, and F. Bullo, “Contracting dynamics for time-varying convex optimization,” *arXiv preprint*, 2023, arXiv:2305.15595.
- [5] A. Simonetto, E. Dall’Anese, S. Paternain, G. Leus, and G. B. Giannakis, “Time-varying convex optimization: Time-structured algorithms and applications,” *Proceedings of the IEEE*, vol. 108, no. 11, pp. 2032–2048, 2020.
- [6] L. Lessard, B. Recht, and A. Packard, “Analysis and design of optimization algorithms via integral quadratic constraints,” *SIAM Journal on Optimization*, vol. 26, no. 1, pp. 57–95, 2016.
- [7] M. Zinkevich, “Online convex programming and generalized infinitesimal gradient ascent,” in *International conference on machine learning*, 2003, pp. 928–936.
- [8] E. Hazan, A. Agarwal, and S. Kale, “Logarithmic regret algorithms for online convex optimization,” *Machine Learning*, vol. 69, no. 2, pp. 169–192, 2007.
- [9] E. Hazan, “Introduction to online convex optimization,” *Foundations and Trends in Optimization*, vol. 2, no. 3–4, pp. 157–325, 2016.
- [10] G. Bianchin and B. V. Scoy, “The discrete-time internal model principle of time-varying optimization: Limitations and algorithm design,” in *IEEE Conf. on Decision and Control*, Dec. 2025, (To appear).
- [11] —, “The internal model principle of time-varying optimization,” *arXiv preprint*, Aug. 2024, arXiv:2407.08037.

- [12] Y. Zhao and M. N. S. Swamy, "A novel technique for tracking time-varying minimum and its applications," in *IEEE Canadian Conference on Electrical and Computer Engineering*, vol. 2, 1998, pp. 910–913.
- [13] M. Fazlyab, S. Paternain, V. M. Preciado, and A. Ribeiro, "Prediction-correction interior-point method for time-varying convex optimization," *IEEE Transactions on Automatic Control*, vol. 63, no. 7, pp. 1973–1986, 2017.
- [14] M. Marchi, J. Bunton, J. P. Silvestre, and P. Tabuada, "A framework for time-varying optimization via derivative estimation," in *European Control Conference*, 2024, pp. 2730–2735.
- [15] A. Simonetto and E. Dall'Anese, "Prediction-correction algorithms for time-varying constrained optimization," *IEEE Transactions on Signal Processing*, vol. 65, no. 20, pp. 5481–5494, 2017.
- [16] A. Hauswirth, Z. He, S. Bolognani, G. Hug, and F. Dörfler, "Optimization algorithms as robust feedback controllers," *Annual Reviews in Control*, vol. 57, p. 100941, 2024.
- [17] N. Bastianello, R. Carli, and S. Zampieri, "Internal model-based online optimization," *IEEE Transactions on Automatic Control*, vol. 69, no. 1, pp. 689–696, 2024.
- [18] U. Casti, N. Bastianello, R. Carli, and S. Zampieri, "A control theoretical approach to online constrained optimization," *Automatica*, vol. 176, p. 112107, 2025.
- [19] A. X. Wu, I. R. Petersen, V. Ugrinovskii, and I. Shames, "An online optimization algorithm for tracking a linearly varying optimal point with zero steady-state error," in *American Control Conference*, 2025, pp. 930–934.
- [20] A. X. Wu, I. R. Petersen, and I. Shames, "A fundamental convergence rate bound for gradient based online optimization algorithms with exact tracking," in *arXiv:2508.21335 [math.OC]*, 2025.
- [21] H. Mohammadi, M. Razaviyayn, and M. R. Jovanović, "Tradeoffs between convergence rate and noise amplification for momentum-based accelerated optimization algorithms," *IEEE Transactions on Automatic Control*, vol. 70, no. 2, pp. 889–904, 2025.
- [22] Y. Nesterov, *Lectures on convex optimization*. Springer, 2018, vol. 137.
- [23] G. Bianchin, J. Cortés, J. I. Poveda, and E. Dall'Anese, "Time-varying optimization of LTI systems via projected primal-dual gradient flows," *IEEE Transactions on Control of Network Systems*, vol. 9, no. 1, pp. 474–486, Mar. 2022.
- [24] A. Hauswirth, S. Bolognani, G. Hug, and F. Dörfler, "Timescale separation in autonomous optimization," *IEEE Transactions on Automatic Control*, vol. 66, no. 2, pp. 611–624, 2021.
- [25] G. Carnevale, N. Mimmo, and G. Notarstefano, "Nonconvex distributed feedback optimization for aggregative cooperative robotics," *Automatica*, vol. 167, p. 111767, 2024.
- [26] A. d'Aspremont, D. Scieur, and A. Taylor, "Acceleration methods," *Foundations and Trends® in Optimization*, vol. 5, no. 1–2, pp. 1–245, 2021.