Asymptotic Mean Ergodicity of Average Consensus Estimators

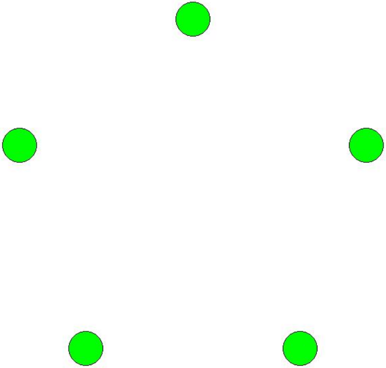
Bryan Van Scoy, Randy A. Freeman, Kevin M. Lynch

Northwestern University

June 6, 2014

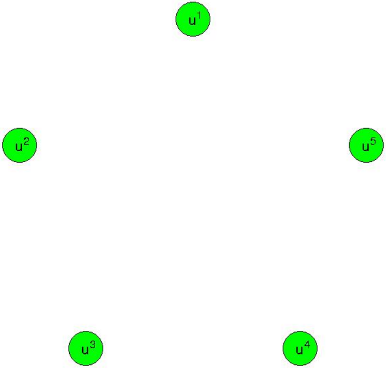
What is average consensus?

- Group of n agents
- Each agent has a local input u^i
- Communication with neighbors represented by directed graph
- Want all agents to be able to calculate the average of all the inputs, $\frac{1}{n}\sum_{i=1}^{n} u^i$



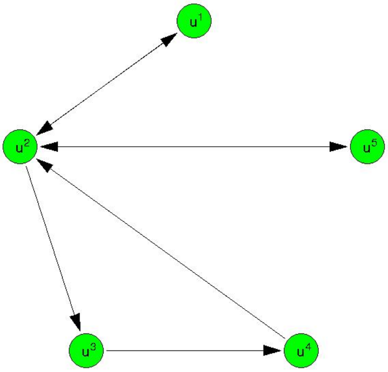
What is average consensus?

- Group of n agents
- Each agent has a local input u^i
- Communication with neighbors represented by directed graph
- Want all agents to be able to calculate the average of all the inputs, $\frac{1}{n}\sum_{i=1}^{n} u^i$



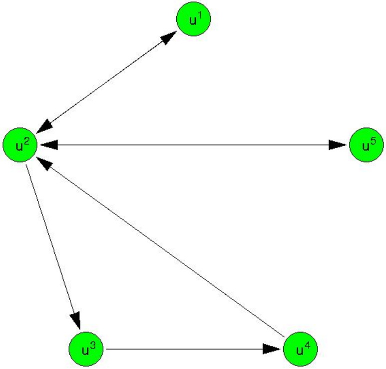
What is average consensus?

- Group of n agents
- Each agent has a local input u^i
- Communication with neighbors represented by directed graph
- Want all agents to be able to calculate the average of all the inputs, $\frac{1}{n}\sum_{i=1}^{n} u^i$



What is average consensus?

- Group of n agents
- Each agent has a local input u^i
- Communication with neighbors represented by directed graph
- Want all agents to be able to calculate the average of all the inputs, $\frac{1}{n}\sum_{i=1}^{n} u^i$

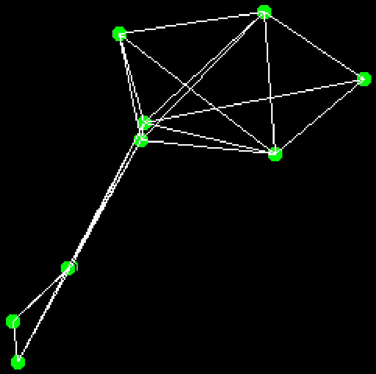


Why average consensus?

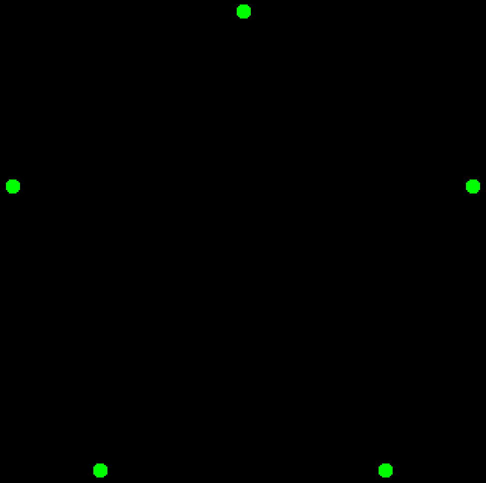
Average consensus is a key building block in many distributed algorithms such as the following:

- Formation control
- Distributed Kalman filtering
- Distributed sensor fusion

Why random switching graphs?



Why random switching graphs?



Assumptions

The graph Laplacian at time k is $L_k \equiv D_k - A_k$ where D_k is the degree matrix and A_k is the adjacency matrix of the graph.

Assumptions

- $\mathrm{E}[L_k]$ balanced and connected
- L_k i.i.d.
- L_k independent of the estimator initial state for all k

Assumptions

The graph Laplacian at time k is $L_k \equiv D_k - A_k$ where D_k is the degree matrix and A_k is the adjacency matrix of the graph.

Assumptions

- $\mathrm{E}[L_k]$ balanced and connected
- L_k i.i.d.
- L_k independent of the estimator initial state for all k

Note

We do not require L_k to be balanced or connected at every time step.

Outline

Initial Condition Estimator

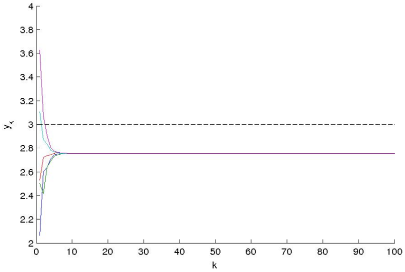
Consider the well-known distributed algorithm

$$x_{k+1}^i = x_k^i - \sum_{j \in \mathcal{N}_i} a_{ij}(x_k^i - x_k^j) \qquad \text{(agent } i\text{)}$$

$$x_{k+1} = x_k - L_k x_k \qquad \text{(vectorized)}$$

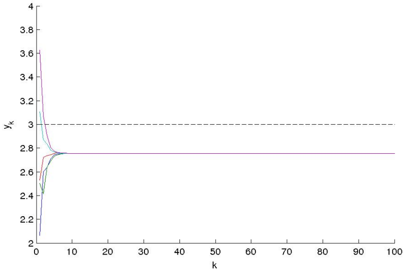
where x_k is the state and L_k is the graph Laplacian at time k, and $x_0 = u$ is the input.

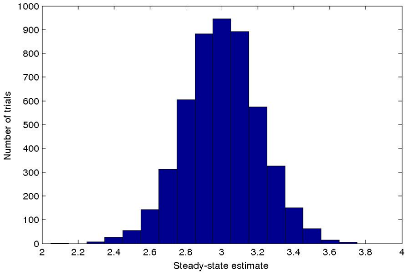
Simulation



- Consensus is achieved.
- Estimate converges to a random variable whose mean is the correct average (Li and Zhang, 2010).
- Average could be approximated by averaging multiple trials.
- This is inefficient...

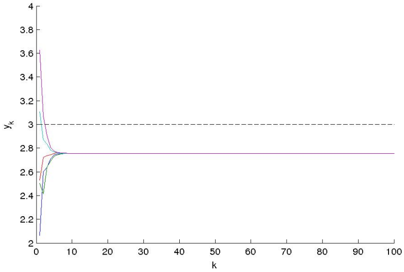
Simulation

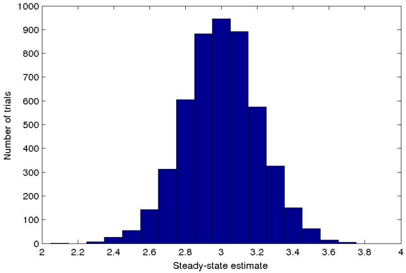




- Consensus is achieved.
- Estimate converges to a random variable whose mean is the correct average (Li and Zhang, 2010).
- Average could be approximated by averaging multiple trials.
- This is inefficient...

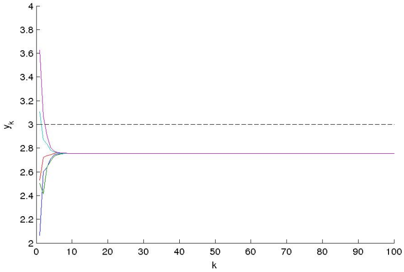
Simulation

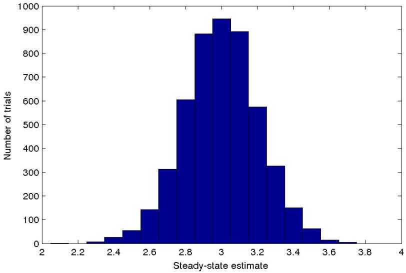




- Consensus is achieved.
- Estimate converges to a random variable whose mean is the correct average (Li and Zhang, 2010).
- Average could be approximated by averaging multiple trials.
- This is inefficient...

Simulation





- Consensus is achieved.
- Estimate converges to a random variable whose mean is the correct average (Li and Zhang, 2010).
- Average could be approximated by averaging multiple trials.
- This is inefficient...

P Estimator

The P estimator equations are

$$x_{k+1} = (1-\gamma)x_k - k_p L_k y_k$$
$$y_k = x_k + u$$

where x_k is the internal state and y_k is the output at time k, and γ and k_p are system parameters.

P Estimator

The P estimator equations are

$$x_{k+1} = (1-\gamma)x_k - k_p L_k y_k$$
$$y_k = x_k + u$$

where x_k is the internal state and y_k is the output at time k, and γ and k_p are system parameters.

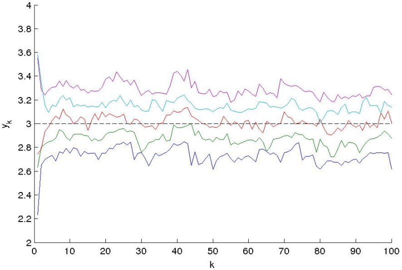
Special case

For $\gamma = 0$ and $k_p = 1$, we have

$$y_{k+1} = y_k - L_k y_k$$

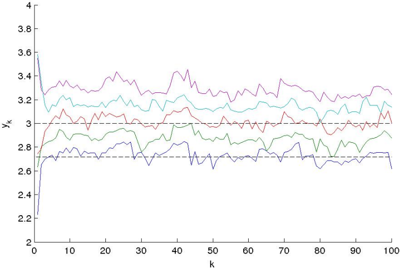
where $y_0 = x_0 + u$.

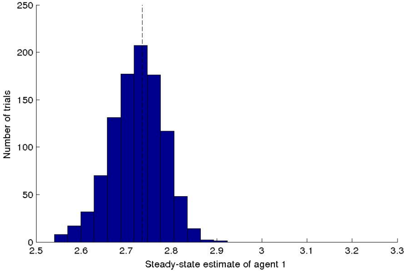
P Estimator ($\gamma \neq 0$)



- Consensus is not achieved.
- The time average of the output converges to the statistical average.
- But the statistical average is not the average of the inputs...

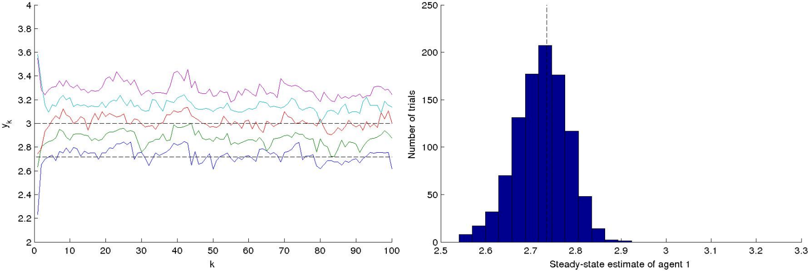
P Estimator ($\gamma \neq 0$)





- Consensus is not achieved.
- The time average of the output converges to the statistical average.
- But the statistical average is not the average of the inputs...

P Estimator ($\gamma \neq 0$)



- Consensus is not achieved.
- The time average of the output converges to the statistical average.
- But the statistical average is not the average of the inputs...

PI Estimator

The PI estimator equations are

$$\begin{aligned} \nu_{k+1} &= (1-\gamma)\nu_k + \gamma u - k_p L_k \nu_k + k_I L_k \eta_k \\ \eta_{k+1} &= \eta_k - k_I L_k \nu_k \\ y_k &= \nu_k \end{aligned}$$

where ν_k and η_k are the internal states at time k and γ, k_p, and k_I are system parameters.

- Convex combination of input and previous state.
- Proportional error term.
- Integral error term.

PI Estimator

The PI estimator equations are

$$\begin{aligned} \nu_{k+1} &= (1-\gamma)\nu_k + \gamma u - k_p L_k \nu_k + k_I L_k \eta_k \\ \eta_{k+1} &= \eta_k - k_I L_k \nu_k \\ y_k &= \nu_k \end{aligned}$$

where ν_k and η_k are the internal states at time k and γ, k_p, and k_I are system parameters.

- Convex combination of input and previous state.
- Proportional error term.
- Integral error term.

PI Estimator

The PI estimator equations are

$$\begin{aligned} \nu_{k+1} &= (1-\gamma)\nu_k + \gamma u - k_p L_k \nu_k + k_I L_k \eta_k \\ \eta_{k+1} &= \eta_k - k_I L_k \nu_k \\ y_k &= \nu_k \end{aligned}$$

where ν_k and η_k are the internal states at time k and γ, k_p, and k_I are system parameters.

- Convex combination of input and previous state.
- Proportional error term.
- Integral error term.

PI Estimator

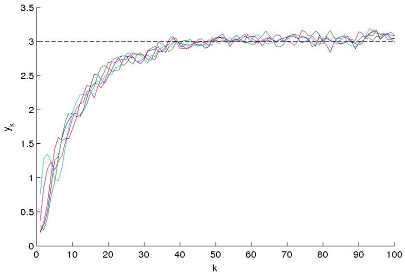
The PI estimator equations are

$$\begin{aligned} \nu_{k+1} &= (1-\gamma)\nu_k + \gamma u - k_p L_k \nu_k + k_I L_k \eta_k \\ \eta_{k+1} &= \eta_k - k_I L_k \nu_k \\ y_k &= \nu_k \end{aligned}$$

where ν_k and η_k are the internal states at time k and γ, k_p, and k_I are system parameters.

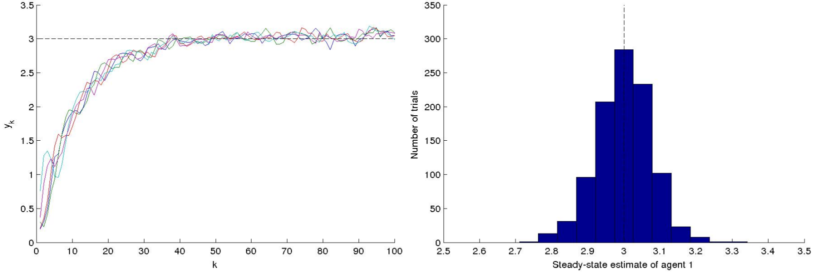
- Convex combination of input and previous state.
- Proportional error term.
- Integral error term.

PI Estimator



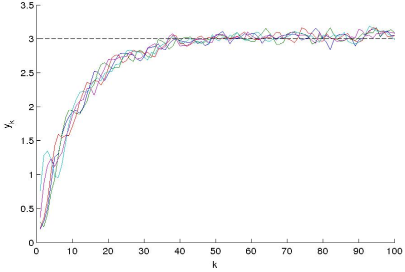
- Consensus is achieved for the time average process.
- The time average of the output converges to the statistical average.
- The statistical average is the average of the inputs, so average consensus is achieved!

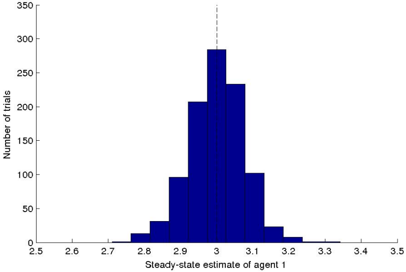
PI Estimator



- Consensus is achieved for the time average process.
- The time average of the output converges to the statistical average.
- The statistical average is the average of the inputs, so average consensus is achieved!

PI Estimator





- Consensus is achieved for the time average process.
- The time average of the output converges to the statistical average.
- The statistical average is the average of the inputs, so average consensus is achieved!

For average consensus, we need

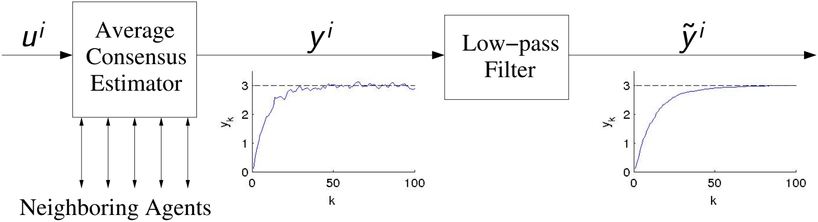
$$\begin{aligned} \text{Time average} &= \text{Statistical average} \quad \text{(ergodicity)} \\ \text{Statistical average} &= \text{Average of inputs} \quad \text{(correctness).} \end{aligned}$$

Then we can low-pass filter the output process to obtain the average of the inputs.

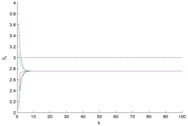
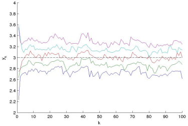
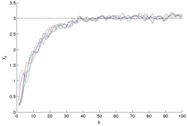
For average consensus, we need

$$\begin{aligned} \text{Time average} &= \text{Statistical average} \quad \text{(ergodicity)} \\ \text{Statistical average} &= \text{Average of inputs} \quad \text{(correctness).} \end{aligned}$$

Then we can low-pass filter the output process to obtain the average of the inputs.



Estimator Properties

Estimator	Ergodic	Correct	
P, $\gamma = 0$	No	Yes[1]	
P, $\gamma \neq 0$	Yes	No	
PI	Yes	Yes	

[1] If the expectation of the initial state is zero.

- **Contribution:** Confirm simulations with analysis
- **Strategy:** Do analysis for a general estimator and apply results to the P and PI estimators

- **Contribution:** Confirm simulations with analysis
- **Strategy:** Do analysis for a general estimator and apply results to the P and PI estimators

Outline

1. Introduction
2. Average Consensus Estimators
   - Initial condition estimator
   - P estimator
   - PI estimator
3. Polynomial Linear Protocol
   - Definition and Examples
   - Separated System
4. Asymptotic Mean Ergodicity and Main Theorem
5. Conclusions

Polynomial Linear Protocol

A polynomial linear protocol (Freeman, Nelson, and Lynch, 2010) of degree ℓ is the collection $\Sigma(L) = [A(L), B(L), C(L), D(L)]$ where

$$A(L) \equiv \sum_{i=0}^{\ell} L^i \otimes A_i \qquad B(L) \equiv \sum_{i=0}^{\ell} L^i \otimes B_i$$

$$C(L) \equiv \sum_{i=0}^{\ell} L^i \otimes C_i \qquad D(L) \equiv \sum_{i=0}^{\ell} L^i \otimes D_i$$

are polynomials in L which describe the linear system

$$\begin{aligned} x_{k+1} &= A(L)x_k + B(L)u_k \\ y_k &= C(L)x_k + D(L)u_k. \end{aligned}$$

Examples

Example 1 (P Estimator)

The P estimator is a polynomial linear protocol of degree one with parameters γ and k_p where

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Examples

Example 1 (P Estimator)

The P estimator is a polynomial linear protocol of degree one with parameters γ and k_p where

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Example 2 (PI Estimator)

The PI estimator is a polynomial linear protocol of degree one with parameters γ, k_p, and k_I where

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Objective

Objective

Want conditions under which the output process y_k of a polynomial linear protocol $\Sigma(L_k)$ is

1. Asymptotically mean ergodic
2. Correct (i.e., the expectation converges to the average of the inputs)

Then the low-pass filtered output converges to the average of the inputs.

Correctness

- A polynomial linear protocol $\Sigma(L_k)$ of degree one is correct if and only if $\Sigma(\mathrm{E}[L_k])$ converges to the average of the inputs.
- This has been characterized (Freeman, Nelson, and Lynch, 2010).
- A necessary condition is A_0 must have an eigenvalue at one.

Correctness

- A polynomial linear protocol $\Sigma(L_k)$ of degree one is correct if and only if $\Sigma(\mathrm{E}[L_k])$ converges to the average of the inputs.
- This has been characterized (Freeman, Nelson, and Lynch, 2010).
- A necessary condition is A_0 must have an eigenvalue at one.

Correctness

- A polynomial linear protocol $\Sigma(L_k)$ of degree one is correct if and only if $\Sigma(\mathrm{E}[L_k])$ converges to the average of the inputs.
- This has been characterized (Freeman, Nelson, and Lynch, 2010).
- A necessary condition is A_0 must have an eigenvalue at one.

Note

- A_0 must have an eigenvalue at one for the system to be correct.
- The Laplacian always has an eigenvalue at zero.
- Therefore, correct systems have an eigenvalue at one.

Note

- A_0 must have an eigenvalue at one for the system to be correct.
- The Laplacian always has an eigenvalue at zero.
- Therefore, correct systems have an eigenvalue at one.

Problem

The steady-state variance of the state could be infinite!

Note

- A_0 must have an eigenvalue at one for the system to be correct.
- The Laplacian always has an eigenvalue at zero.
- Therefore, correct systems have an eigenvalue at one.

Problem

The steady-state variance of the state could be infinite!

Solution

The state corresponding to the eigenvalue at one must be unobservable.

Separated System

Definition 3 (Reduced Laplacian)

The reduced Laplacian $\hat{L}$ is defined as $\hat{L} := S^T L S$ where $Q = \begin{bmatrix} v & S \end{bmatrix} \in \mathbb{R}^{n \times n}$ is orthogonal and $v = 1_n/\sqrt{n}$.

Separated System

Definition 3 (Reduced Laplacian)

The reduced Laplacian $\hat{L}$ is defined as $\hat{L} := S^T L S$ where $Q = \begin{bmatrix} v & S \end{bmatrix} \in \mathbb{R}^{n \times n}$ is orthogonal and $v = 1_n/\sqrt{n}$.

Performing the change of variable $\tilde{x}_k = (Q \otimes I)^T x_k$, the separated system $\tilde{\Sigma}(L)$ is

$$\tilde{A}(L) = \begin{bmatrix} A_0 & (v \otimes I)^T A(L)(S \otimes I) \\ 0 & A(\hat{L}) \end{bmatrix} \quad \tilde{B}(L) = \begin{bmatrix} (v \otimes I)^T B(L) \\ (S \otimes I)^T B(L) \end{bmatrix}$$

$$\tilde{C}(L) = \begin{bmatrix} v \otimes C_0 & C(L)(S \otimes I) \end{bmatrix} \qquad \tilde{D}(L) = D(L).$$

Outline

Definition 4 (Asymptotically Wide-Sense Stationary)

The process X_k is asymptotically wide-sense stationary if and only if the mean and covariance of the steady-state process do not change with time; that is, the limits

$$m_X \equiv \lim_{n\to\infty} E[X_n] \quad \text{and} \quad C_X(k) \equiv \lim_{n\to\infty} \mathrm{COV}[X_{k+n}, X_n]$$

exist and are finite where m_X is the mean and $C_X(k)$ is the covariance of the steady-state process.

Theorem 5 (Asymptotic Mean Ergodicity)

Let $\{X_k\}_{k=1}^{\infty}$ be a single-sided asymptotically wide-sense stationary discrete-time random process with limiting mean m_X and limiting covariance $C_X(k)$. The process is asymptotically mean ergodic, that is,

$$\lim_{T\to\infty}\lim_{n\to\infty}\frac{1}{T}\sum_{k=0}^{T-1}X_{k+n}=m_X$$

in the mean square sense if and only if

$$\lim_{T\to\infty}\frac{1}{T}\sum_{k=-(T-1)}^{T-1}\left(1-\frac{|k|}{T}\right)C_X(k)=0$$

(similar to a result in (Leon-Garcia, 2008)).

Corollary 6

An asymptotically wide-sense stationary random process X_k with steady-state covariance given by

$$C_X(k) = \lambda^{|k|}$$

is asymptotically mean ergodic if and only if $|\lambda| \leq 1$ and $\lambda \neq 1$.

Corollary 6

An asymptotically wide-sense stationary random process X_k with steady-state covariance given by

$$C_X(k) = \lambda^{|k|}$$

is asymptotically mean ergodic if and only if $|\lambda| \leq 1$ and $\lambda \neq 1$.

Corollary 7

An asymptotically wide-sense stationary random process X_k with steady-state covariance given by

$$C_X(k) = CA^{|k|}B$$

where $A \in \mathbb{R}^{n \times n}$ is convergent, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$, and any eigenvalue of A at one is either uncontrollable through B or unobservable through C, is asymptotically mean ergodic.

Main Theorem

Theorem 8 (Asymptotically Mean Ergodic)

Consider the time-varying polynomial linear protocol $\Sigma(L_k)$ of degree ℓ based on the time-varying Laplacian L_k where $E[L_k]$ is balanced and connected, and L_k are i.i.d. and independent of the initial state for all k. The output process due to a constant input is asymptotically mean ergodic if the following hold:

1. *A_0 is convergent,*
2. *any eigenvalues of A_0 at one are unobservable through C_0,*
3. *$\rho\big(E\big[A(\hat{L}_k)\big]\big) < 1$, and*
4. *$C_i = D_i = 0$ for $1 \leq i \leq \ell$.*

Outline

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1 - \cancelto{0}{\gamma} & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 1: $\gamma = 0$

- ✓ A_0 is convergent
- ✗ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct (if the expectation of the initial state is zero)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1 - \cancelto{0}{\gamma} & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 1: $\gamma = 0$

- ✓ A_0 is convergent
- ✗ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct (if the expectation of the initial state is zero)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1 - \cancelto{0}{\gamma} & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 1: $\gamma = 0$

- ✓ A_0 is convergent
- ✗ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct (if the expectation of the initial state is zero)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1 - \cancelto{0}{\gamma} & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 1: $\gamma = 0$

- ✓ A_0 is convergent
- ✗ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct (if the expectation of the initial state is zero)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1 - \cancelto{0}{\gamma} & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 1: $\gamma = 0$

- ✓ A_0 is convergent
- ✗ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\big(\mathrm{E}\big[A(\hat{L}_k)\big]\big) < 1$ (for appropriate k_p)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct (if the expectation of the initial state is zero)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L)\end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\cancelto{0}{\gamma} & 0 \\ \hline 1 & 0\end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0\end{array}\right]$$

Case 1: $\gamma = 0$

- ✓ A_0 is convergent
- ✗ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\left(\mathrm{E}\left[A(\hat{L}_k)\right]\right) < 1$ (for appropriate k_p)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct (if the expectation of the initial state is zero)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 2: $\gamma \neq 0$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✗ Correct (need A_0 to have an eigenvalue at one)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 2: $\gamma \neq 0$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✗ Correct (need A_0 to have an eigenvalue at one)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 2: $\gamma \neq 0$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✗ Correct (need A_0 to have an eigenvalue at one)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 2: $\gamma \neq 0$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✗ Correct (need A_0 to have an eigenvalue at one)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 2: $\gamma \neq 0$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\big(\mathrm{E}\big[A(\hat{L}_k)\big]\big) < 1$ (for appropriate k_p, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✗ Correct (need A_0 to have an eigenvalue at one)

P Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{c|c} 1-\gamma & 0 \\ \hline 1 & 0 \end{array}\right] + L \otimes \left[\begin{array}{c|c} -k_p & -k_p \\ \hline 0 & 0 \end{array}\right]$$

Case 2: $\gamma \neq 0$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\big(\mathrm{E}\big[A(\hat{L}_k)\big]\big) < 1$ (for appropriate k_p, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✗ Correct (need A_0 to have an eigenvalue at one)

PI Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, k_I, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct

PI Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, k_I, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct

PI Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\left(\mathrm{E}\left[A(\hat{L}_k)\right]\right) < 1$ (for appropriate k_p, k_I, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct

PI Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

✓ A_0 is convergent

✓ any eigenvalues of A_0 at one are unobservable through C_0

✓ $\rho(\mathrm{E}[A(\hat{L}_k)]) < 1$ (for appropriate k_p, k_I, γ)

✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$

✓ Correct

PI Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

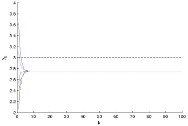
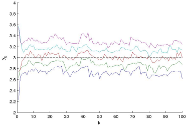
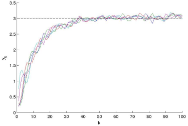
- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\big(\mathrm{E}\big[A(\hat{L}_k)\big]\big) < 1$ (for appropriate k_p, k_I, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct

PI Estimator

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right] = I \otimes \left[\begin{array}{cc|c} 1-\gamma & 0 & \gamma \\ 0 & 1 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] + L \otimes \left[\begin{array}{cc|c} -k_p & k_I & 0 \\ -k_I & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right]$$

- ✓ A_0 is convergent
- ✓ any eigenvalues of A_0 at one are unobservable through C_0
- ✓ $\rho\big(\mathrm{E}\big[A(\hat{L}_k)\big]\big) < 1$ (for appropriate k_p, k_I, γ)
- ✓ $C_i = D_i = 0$ for $1 \leq i \leq \ell$
- ✓ Correct

Estimator Properties

Estimator	Ergodic	Correct	
P, $\gamma = 0$	No	Yes[1]	
P, $\gamma \neq 0$	Yes	No	
PI	Yes	Yes	

[1] If the expectation of the initial state is zero.

Summary

- Defined asymptotic mean ergodicity and gave an ergodic theorem.
- Characterized the asymptotic mean ergodicity property for polynomial linear protocols.
- Applied results to the P and PI estimators to explain behavior over i.i.d. random graphs.

Summary

- Defined asymptotic mean ergodicity and gave an ergodic theorem.
- Characterized the asymptotic mean ergodicity property for polynomial linear protocols.
- Applied results to the P and PI estimators to explain behavior over i.i.d. random graphs.

Summary

- Defined asymptotic mean ergodicity and gave an ergodic theorem.
- Characterized the asymptotic mean ergodicity property for polynomial linear protocols.
- Applied results to the P and PI estimators to explain behavior over i.i.d. random graphs.

References

Cai, Kai and H. Ishii (2012). "Average Consensus on Arbitrary Strongly Connected Digraphs with Dynamic Topologies". In: *Proceedings of the 2012 American Control Conference*, pp. 14–19.

Chen, Yin et al. (2010). "Corrective Consensus: Converging to the Exact Average". In: *Proceedings of the 49th IEEE Conference on Decision and Control*, pp. 1221–1228. DOI: 10.1109/CDC.2010.5717925.

Cortes, J. (2009). "Distributed Kriged Kalman Filter for Spatial Estimation". In: *IEEE Transactions on Automatic Control* 54.12, pp. 2816–2827. ISSN: 0018-9286. DOI: 10.1109/TAC.2009.2034192.

Freeman, R.A., T.R. Nelson, and K.M. Lynch (2010). "A Complete Characterization of a Class of Robust Linear Average Consensus Protocols". In: *Proceedings of the 2010 American Control Conference*, pp. 3198–3203.

Freeman, R.A., Peng Yang, and K.M. Lynch (2006). "Stability and Convergence Properties of Dynamic Average Consensus Estimators". In: *Proceedings of the 45th IEEE Conference on Decision and Control*, pp. 338–343. DOI: 10.1109/CDC.2006.377078.

Leon-Garcia, A. (2008). *Probability, Statistics, and Random Processes for Electrical Engineering*. Pearson/Prentice Hall.

Li, Tao and Ji-Feng Zhang (2010). "Consensus Conditions of Multi-Agent Systems With Time-Varying Topologies and Stochastic Communication Noises". In: *IEEE Transactions on Automatic Control* 55.9, pp. 2043–2057. ISSN: 0018-9286. DOI: 10.1109/TAC.2010.2042982.

Peterson, Cameron K. and Derek A. Paley (2013). "Distributed Estimation for Motion Coordination in an Unknown Spatially Varying Flowfield". In: *Journal of Guidance, Control, and Dynamics* 36.3, pp. 894–898. ISSN: 0731-5090. DOI: 10.2514/1.59453.

Vaidya, N.H., C.N. Hadjicostis, and A.D. Dominguez-Garcia (2012). "Robust Average Consensus over Packet Dropping Links: Analysis via Coefficients of Ergodicity". In: *Proceedings of the 51st IEEE Conference on Decision and Control*, pp. 2761–2766. DOI: 10.1109/CDC.2012.6426252.

Yang, Peng, R.A. Freeman, and K.M. Lynch (2008). "Multi-Agent Coordination by Decentralized Estimation and Control". In: *IEEE Transactions on Automatic Control* 53.11, pp. 2480–2496. ISSN: 0018-9286. DOI: 10.1109/TAC.2008.2006925.