Optimal Worst-Case Dynamic Average Consensus

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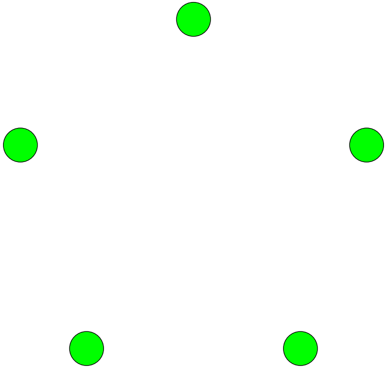
Outline

What is average consensus?

- Group of N agents
- Each agent has a local input u^i
- Communication with neighbors represented by an undirected graph
- Want all agents to calculate the average of all the inputs, $\frac{1}{N}\sum_{i=1}^{N} u^i$



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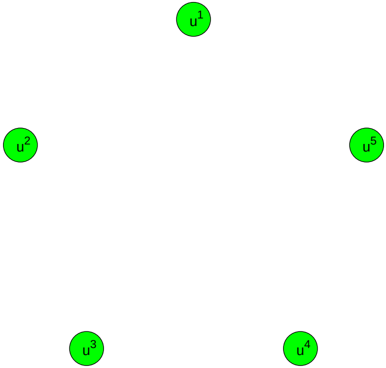

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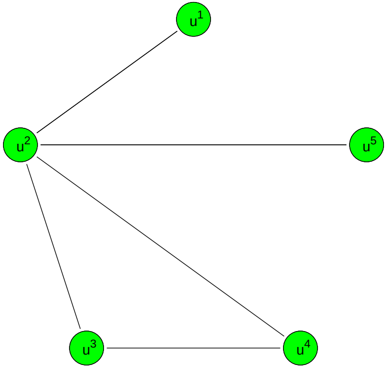

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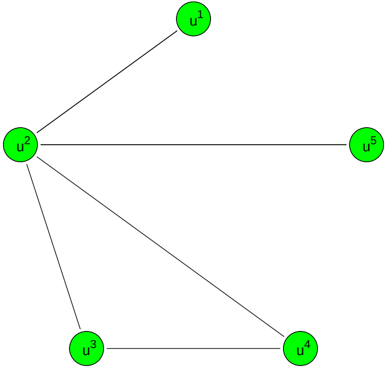

Applications of average consensus

Average consensus is a key building block in many distributed algorithms such as the following:

- Formation control (Yang, Freeman, and Lynch, 2008)
- Distributed merging of feature-based maps (Aragues, Cortes, and Sagues, 2012)
- Distributed environmental monitoring (Bai, Freeman, and Lynch; Cortes; Olfati-Saber; Peterson and Paley, 2011; 2009; 2005; 2013)

Goal

Design average consensus estimators which are:

- simple
- scalable
- robust
  - to initial conditions
  - to changes in graph topology
- accurate
- internally stable
- able to track dynamic signals
- fast (asymptotic convergence factor)

Outline

Initial condition estimator

Agent i:

$$\underbrace{x^i_{k+1}}_{\substack{\text{new}\\\text{estimate}}} = \underbrace{x^i_k}_{\substack{\text{current}\\\text{estimate}}} - \underbrace{k_p \sum_{j \in \mathcal{N}_i} a_{ij}(x^i_k - x^j_k)}_{\substack{\text{weighted sum of estimate}\\\text{differences among neighbors}}} \qquad x^i_0 = u^i$$

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Full system:

$$x_{k+1} = \underbrace{(I - k_p L)}_{W} x_k \qquad x_0 = u$$

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This is referred to as *static* consensus since the input appears in the initial condition rather than the update equations.

Static consensus

Assumption 1

- *The graph is connected and undirected.*
- *All non-zero eigenvalues of L are in the interval $[\lambda_{min}, \lambda_{max}]$.*

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Then the worst-case asymptotic convergence factor is

$$\alpha = \max_{\lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]} |1 - k_p \lambda|.$$

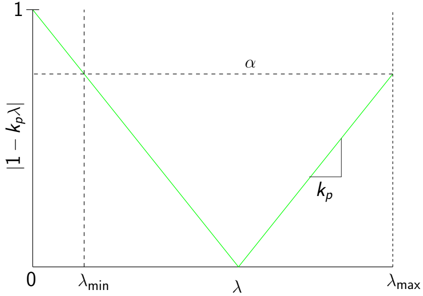
Static consensus design

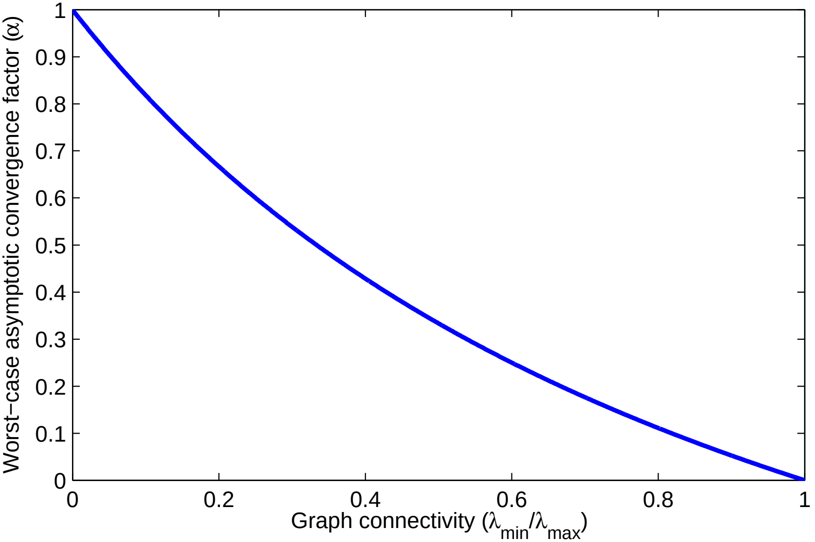
Choose k_p to minimize the asymptotic convergence factor:

$$\begin{aligned}\alpha^* &= \min_{k_p} \alpha \\ &= \min_{k_p} \max_{\lambda \in [\lambda_{\min}, \lambda_{\max}]} |1 - k_p \lambda| \\ &= \frac{1 - \lambda_{\min}/\lambda_{\max}}{1 + \lambda_{\min}/\lambda_{\max}}\end{aligned}$$

where

$$k_p = \frac{2}{\lambda_{\min} + \lambda_{\max}}.$$




Worst−case asymptotic convergence factor (α)
1
0.9
0.8
0.7
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Graph connectivity (λmin/λmax)

Outline

Static vs. Dynamic

Static

$$x_{k+1} = (I - k_p L)x_k, \quad x_0 = u$$
$$y_k = x_k \qquad \text{(estimate of the average of } u\text{)}$$

Static vs. Dynamic

Static

$$\begin{aligned} x_{k+1} &= (I - k_p L) x_k, \quad x_0 = u \\ y_k &= x_k \qquad \text{(estimate of the average of } u\text{)} \end{aligned}$$

Dynamic (non-robust)

$$\begin{aligned} x_{k+1} &= (I - k_p L) x_k - k_p L u_k, \quad x_0 = 0 \\ y_k &= x_k + u_k \end{aligned}$$

Static vs. Dynamic

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Dynamic (robust)

$$x_{k+1} = A(L)x_k + B(L)u_k, \quad x_0 = \text{anything}$$
$$y_k = C(L)x_k + D(L)u_k$$

where $A(L), B(L), C(L), D(L)$ satisfy certain properties (Freeman, Nelson, and Lynch, 2010).

Separated System

Full system:

$$\left[\begin{array}{c|c} A(L) & B(L) \\ \hline C(L) & D(L) \end{array}\right]$$

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Separated system (unknown graph):

$$\left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right], \quad \lambda \in \{0\} \cup [\lambda_{\min}, \lambda_{\max}]$$

Problem 1

Determine conditions on $A(\lambda)$ such that $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ can be chosen such that the estimator

- *achieves exact average consensus for constant inputs and*
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Problem 2

Design an estimator which minimizes the worst-case asymptotic convergence factor over all graphs with $\mathrm{eig}(L) \subseteq \Lambda$ *and has the properties listed in Problem 1. That is, solve*

$$\alpha = \min_{A_i} \max_{\lambda \in \Lambda} \rho(A(\lambda)) \quad \textit{subject to conditions from Problem 1.}$$

Condition 1 (Condition on $A(\lambda) = A_0 + \lambda A_1$)

There exist $v, w, p_0, p_1, q_0, q_1 \in \mathbb{R}^n$ *such that*

$$0 = \begin{bmatrix} I - A_0 & \\ -A_1 & I - A_0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_0 - v \end{bmatrix}$$

$$0 = \begin{bmatrix} I - A_0^T & \\ -A_1^T & I - A_0^T \end{bmatrix} \begin{bmatrix} q_1 \\ q_0 - w \end{bmatrix}$$

$$1 = (q_0 - w)^T (I - A_0)(p_0 - v).$$

Main Theorem

Theorem 3

Consider an estimator with $A(\lambda)$ convergent for all $\lambda \in \mathrm{eig}(L)$.

1. *(Necessity) If the estimator achieves average consensus for constant inputs and is robust to initial conditions, then $A(\lambda)$ satisfies Condition 1.*
2. *(Sufficiency) If $A(\lambda)$ satisfies Condition 1, then there exist $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ such that the estimator achieves average consensus for constant inputs and is robust to initial conditions.*

Estimator Structure

For an estimator which satisfies Condition 1,

$$\left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right] = \left[\begin{array}{c|c} A_0 & (I - A_0)v \\ \hline w^T(I - A_0) & 1 - w^T(I - A_0)v \end{array}\right] + \lambda \left[\begin{array}{c|c} A_1 & -A_1 p_0 \\ \hline -q_0{}^T A_1 & -q_0{}^T A_1 p_0 \end{array}\right]$$

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Corollary 4 (Van Scoy, Freeman, and Lynch, 2014)

The above estimator is asymptotically mean ergodic if $q_0 = 0$.

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The above estimator is asymptotically mean ergodic if $q_0 = 0$.

Corollary 5 (Zhu and Martinez, 2010 and Kia, Cortes, and Martinez, 2013)

The above estimator is cascadable if $v = 0$.

Estimator Structure

For an estimator which satisfies Condition 1 and is asymptotically mean ergodic and cascadable,

$$\left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right] = \left[\begin{array}{c|c} A_0 & 0 \\ \hline w^T(I - A_0) & 1 \end{array}\right] + \lambda \left[\begin{array}{c|c} A_1 & -A_1 p_0 \\ \hline 0 & 0 \end{array}\right].$$

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Now choose A_0 and A_1 to minimize the worst-case asymptotic convergence factor subject to Condition 1.

Outline

Robust Optimization of Spectral Radius

Solve: $\alpha = \min_{A_i} \max_{\lambda \in \Lambda} \rho(A(\lambda))$ subject to Condition 1

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Let $\bar{\lambda}$ be fixed. Then $\rho(A(\bar{\lambda})) < \alpha$ if and only if $H > 0$ where H is a matrix whose coefficients are polynomials in A_0 and A_1.

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Lemma 7 (Chesi, 2013)

$\rho(A(\lambda)) < \alpha$ for all $\lambda \in [\lambda_{min}, \lambda_{max}]$ if and only if the following conditions hold:

- *$\rho(A(\bar{\lambda})) < \alpha$ for some $\bar{\lambda} \in [\lambda_{min}, \lambda_{max}]$;*
- *several matrix inequalities are satisfied which are polynomials in A_0 and A_1.*

Polynomial Matrix Inequality (PMI)

Solutions to PMIs (Henrion and Lasserre, 2006):

- Solve convex LMI relaxations
- Solution converges to the global optimum as size of relaxation increases
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For $A_0, A_1 \in \mathbb{R}^{2\times 2}$, the conditions can be expressed as LMIs.

Outline

Problem 1

To satisfy Condition 1, we can use the parameterization

$$A(\lambda) = \begin{bmatrix} 1-\gamma & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} -k_p & k_I \\ -1 & 0 \end{bmatrix}$$

without loss of generality which gives

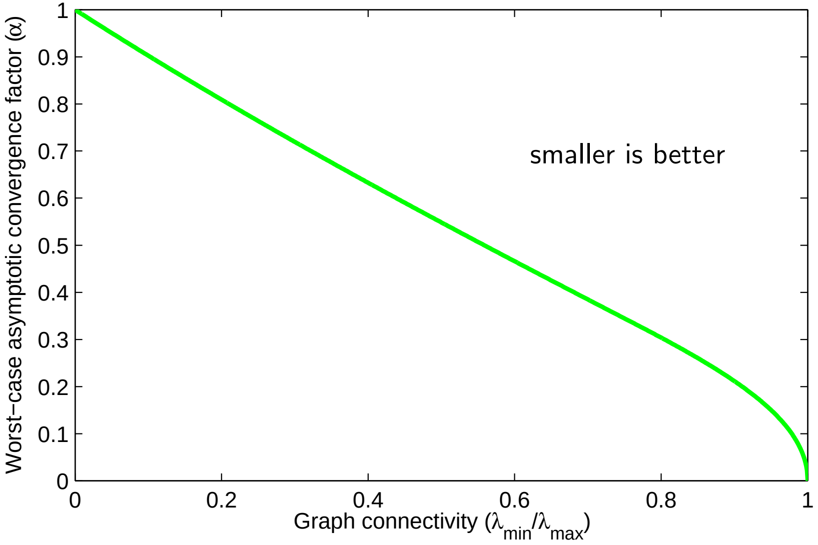
$$\begin{bmatrix} p_1 \\ p_0 - v \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma/k_I \\ -1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} q_1 \\ q_0 - w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1/\gamma \\ 0 \end{bmatrix}.$$

Choose $v = q_0 = 0$ for the estimator to be asymptotically mean ergodic and cascadable.

Estimator Form

$$\left[\begin{array}{c|c} A(\lambda) & B(\lambda) \\ \hline C(\lambda) & D(\lambda) \end{array}\right] = \left[\begin{array}{cc|c} 1-\gamma & 0 & 0 \\ 0 & 1 & 0 \\ \hline 1 & 0 & 1 \end{array}\right] + \lambda \left[\begin{array}{cc|c} -k_p & k_I & -k_p \\ -1 & 0 & -1 \\ \hline 0 & 0 & 0 \end{array}\right]$$

Choose γ, k_p, and k_I to minimize the worst-case asymptotic convergence factor of $A(\lambda)$.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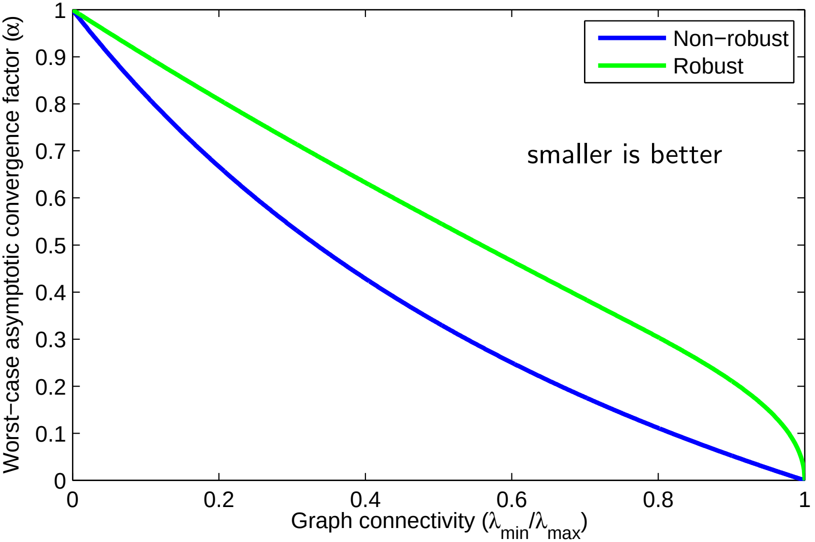
Graph connectivity (λmin/λmax)
smaller is better


Non−robust
Robust
smaller is better
Worst−case asymptotic convergence factor (α)
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0.3
0.4
0.5
0.6
0.7
0.8
0.9
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General design procedure

In general,

- $\alpha = \alpha(\lambda_{\text{min}}/\lambda_{\text{max}})$,
- α is monotonically non-increasing.

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- α is monotonically non-increasing.

General design procedure:

1. Design the weighted Laplacian to maximize the ratio $\lambda_{\text{min}}/\lambda_{\text{max}}$.
2. Design the estimator to have the desired properties:
   - exact
   - robust to initial conditions
   - asymptotically mean ergodic
   - asymptotic convergence factor

Conclusions

To conclude, we have:

- characterized the structure of estimators which are exact and robust to initial conditions,
- used this structure to minimize the worst-case asymptotic convergence factor for a two-dimensional estimator, and
- setup the problem of designing higher dimensional estimators as a set of PMIs. Techniques exist for finding the global optimal solution, although they are computationally difficult.

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Closed-Form Solution

The LMIs in Lemma 7 can be solved in closed-form to obtain

$$\alpha = \begin{cases} \dfrac{\lambda_r^2 - 8\lambda_r + 8}{8 - \lambda_r^2}, & 0 < \lambda_r \le 3 - \sqrt{5} \\ \dfrac{\sqrt{(1-\lambda_r)(5\lambda_r^2 - \lambda_r^3 + 4)} - \lambda_r(1-\lambda_r)}{2(\lambda_r^2 + 1)}, & 3 - \sqrt{5} < \lambda_r \le 1 \end{cases}$$

where $\lambda_r = \lambda_{\min}/\lambda_{\max}$ and the estimator parameters are

$$\gamma = 1 - \alpha, \qquad k_p = \frac{1}{\lambda_{\max}} \frac{\alpha(1-\alpha)\lambda_r}{\alpha + \lambda_r - 1}, \qquad k_I = \frac{1-\alpha}{\lambda_{\min}}.$$