Optimal Worst-Case Dynamic Average Consensus

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Outline

1. Introduction
2. Motivation: Static consensus
3. Problem Setup
4. Robust Optimization of Spectral Radius
5. Example: 2-state estimator
What is average consensus?

- **Group of** $N$ **agents**
- Each agent has a local input $u^i$
- Communication with neighbors represented by an undirected graph
- Want all agents to calculate the average of all the inputs,

$$\frac{1}{N} \sum_{i=1}^{N} u^i$$
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Applications of average consensus

Average consensus is a key building block in many distributed algorithms such as the following:

- Formation control (Yang, Freeman, and Lynch, 2008)
- Distributed merging of feature-based maps (Aragues, Cortes, and Sagues, 2012)
- Distributed environmental monitoring (Bai, Freeman, and Lynch; Cortes; Olfati-Saber; Peterson and Paley, 2011; 2009; 2005; 2013)
Goal

Design average consensus estimators which are:

- simple
- scalable
- robust
  - to initial conditions
  - to changes in graph topology
- accurate
- internally stable
- able to track dynamic signals
- fast (asymptotic convergence factor)
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3. Problem Setup
4. Robust Optimization of Spectral Radius
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Initial condition estimator

Agent $i$:

\[
\hat{x}^{i}_{k+1} = x^{i}_{k} - k_p \sum_{j \in N_i} a_{ij}(x^{i}_{k} - x^{j}_{k}) \\
\]

new estimate

current estimate

weighted sum of estimate differences among neighbors

\[
x^{i}_{0} = u^{i}
\]
Initial condition estimator

Agent $i$:

\[
\begin{align*}
\hat{x}_{k+1}^i &= x_k^i - k_p \sum_{j \in N_i} a_{ij} (x_k^i - x_k^j) \\
\text{new estimate} &= \text{current estimate} - \text{weighted sum of estimate differences among neighbors}
\end{align*}
\]

Full system:

\[
\begin{align*}
x_{k+1} &= (I - k_p L) x_k \\
x_0 &= u
\end{align*}
\]
Initial condition estimator

Agent $i$:

$$x_{k+1}^i = x_k^i - k_p \sum_{j \in N_i} a_{ij} (x_k^i - x_k^j)$$

$$x_0^i = u^i$$

Full system:

$$x_{k+1} = W(L - k_p L)x_k$$

$$x_0 = u$$

This is referred to as static consensus since the input appears in the initial condition rather than the update equations.
Static consensus

Assumption 1

- The graph is connected and undirected.
- All non-zero eigenvalues of L are in the interval $[\lambda_{\text{min}}, \lambda_{\text{max}}]$. 

$x_{k+1} = (I - k_{\text{opt}}L)x_k, x_0 = u$ 

Then the worst-case asymptotic convergence factor is $\alpha = \max_{\lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]} |1 - k_{\text{opt}}\lambda|$. 

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Optimal Worst-Case Dynamic Average Consensus
Static consensus

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$x_{k+1} = (I - k_p L)x_k, \quad x_0 = u$
Introduction
Motivation: Static consensus
Problem Setup
Robust Optimization of Spectral Radius
Example: 2-state estimator

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- All non-zero eigenvalues of $L$ are in the interval $[\lambda_{\text{min}}, \lambda_{\text{max}}]$.

\[ x_{k+1} = (I - k_p L) x_k, \quad x_0 = u \]

Then the worst-case asymptotic convergence factor is

\[ \alpha = \max_{\lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}]} |1 - k_p \lambda|. \]
Choose $k_p$ to minimize the asymptotic convergence factor:

$$\alpha^* = \min_{k_p} \alpha$$

$$= \min_{k_p} \max_{\lambda \in [\lambda_{\min}, \lambda_{\max}]} |1 - k_p \lambda|$$

$$= \frac{1 - \lambda_{\min}/\lambda_{\max}}{1 + \lambda_{\min}/\lambda_{\max}}$$

where

$$k_p = \frac{2}{\lambda_{\min} + \lambda_{\max}}.$$
Introduction
Motivation: Static consensus
Problem Setup
Robust Optimization of Spectral Radius
Example: 2-state estimator

Graph connectivity ($\lambda_{\text{min}} / \lambda_{\text{max}}$)
Worst-case asymptotic convergence factor ($\alpha$)

smaller is better

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Optimal Worst-Case Dynamic Average Consensus
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3. Problem Setup
4. Robust Optimization of Spectral Radius
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Static vs. Dynamic

Static

\[ x_{k+1} = (I - k_p L)x_k, \quad x_0 = u \]
\[ y_k = x_k \quad \text{(estimate of the average of } u) \]
Static vs. Dynamic

Static

\[ x_{k+1} = (I - k_p L)x_k, \quad x_0 = u \]

\[ y_k = x_k \quad \text{(estimate of the average of } u) \]

Dynamic (non-robust)

\[ x_{k+1} = (I - k_p L)x_k - k_p Lu_k, \quad x_0 = 0 \]

\[ y_k = x_k + u_k \]
Static vs. Dynamic

Static

\[ x_{k+1} = (I - k_p L)x_k, \quad x_0 = u \]
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Dynamic (non-robust)

\[ x_{k+1} = (I - k_p L)x_k - k_p Lu_k, \quad x_0 = 0 \]
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Dynamic (robust)

\[ x_{k+1} = A(L)x_k + B(L)u_k, \quad x_0 = \text{anything} \]
\[ y_k = C(L)x_k + D(L)u_k \]

where \( A(L), B(L), C(L), D(L) \) satisfy certain properties (Freeman, Nelson, and Lynch, 2010).
Introduction
Motivation: Static consensus

Problem Setup

Robust Optimization of Spectral Radius

Example: 2-state estimator

Separated System

Full system:

\[
\begin{bmatrix}
A(L) & B(L) \\
C(L) & D(L)
\end{bmatrix}
\]
Separated System

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Separated system:

\[
\begin{bmatrix}
A(\lambda_i) & B(\lambda_i) \\
C(\lambda_i) & D(\lambda_i)
\end{bmatrix}, \quad \lambda_i \in \text{eig}(L)
\]
Separated System

Full system:

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C(\lambda_i) & D(\lambda_i)
\end{bmatrix}, \quad \lambda_i \in \text{eig}(L)
\]

Separated system (unknown graph):

\[
\begin{bmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix}, \quad \lambda \in \{0\} \cup [\lambda_{\min}, \lambda_{\max}]
\]
Problem 1

Determine conditions on $A(\lambda)$ such that $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ can be chosen such that the estimator

- achieves exact average consensus for constant inputs and
- is robust to initial conditions.
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- achieves exact average consensus for constant inputs and
- is robust to initial conditions.

Problem 2

Design an estimator which minimizes the worst-case asymptotic convergence factor over all graphs with $\text{eig}(L) \subseteq \Lambda$ and has the properties listed in Problem 1. That is, solve

$$\alpha = \min_{A_i} \max_{\lambda \in \Lambda} \rho(A(\lambda)) \quad \text{subject to conditions from Problem 1.}$$
Condition 1 (Condition on $A(\lambda) = A_0 + \lambda A_1$)

There exist $v, w, p_0, p_1, q_0, q_1 \in \mathbb{R}^n$ such that

\[
0 = \begin{bmatrix}
    I - A_0 \\
    -A_1 & I - A_0
\end{bmatrix}
\begin{bmatrix}
    p_1 \\
    p_0 - v
\end{bmatrix}
\]

\[
0 = \begin{bmatrix}
    I - A_0^T \\
    -A_1^T & I - A_0^T
\end{bmatrix}
\begin{bmatrix}
    q_1 \\
    q_0 - w
\end{bmatrix}
\]

\[
1 = (q_0 - w)^T (I - A_0) (p_0 - v).
\]
Consider an estimator with $A(\lambda)$ convergent for all $\lambda \in \text{eig}(L)$.

1. (Necessity) If the estimator achieves average consensus for constant inputs and is robust to initial conditions, then $A(\lambda)$ satisfies Condition 1.

2. (Sufficiency) If $A(\lambda)$ satisfies Condition 1, then there exist $B(\lambda)$, $C(\lambda)$, and $D(\lambda)$ such that the estimator achieves average consensus for constant inputs and is robust to initial conditions.
Estimator Structure

For an estimator which satisfies Condition 1,

\[
\begin{bmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix}
= \begin{bmatrix}
A_0 & (I - A_0)v \\
w^T(I - A_0) & 1 - w^T(I - A_0)v
\end{bmatrix}
+ \lambda \begin{bmatrix}
A_1 & -A_1p_0 \\
-q_0^TA_1 & -q_0^TA_1p_0
\end{bmatrix}
\]

Corollary 4 (Van Scoy, Freeman, and Lynch, 2014)
The above estimator is asymptotically mean ergodic if \( q_0 = 0 \).

Corollary 5 (Zhu and Martinez, 2010 and Kia, Cortes, and Martinez, 2013)
The above estimator is cascadable if \( v = 0 \).
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\begin{bmatrix}
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= \begin{bmatrix}
A_0 & (I - A_0) v \\
\frac{w^T(I - A_0)}{w^T(I - A_0)} & 1 - \frac{w^T(I - A_0)}{w^T(I - A_0)} v
\end{bmatrix}
\]

\[+ \lambda \begin{bmatrix}
A_1 & -A_1 p_0 \\
-q_0^T A_1 & -q_0^T A_1 p_0
\end{bmatrix}\]

**Corollary 4 (Van Scoy, Freeman, and Lynch, 2014)**

*The above estimator is asymptotically mean ergodic if \(q_0 = 0\).*

**Corollary 5 (Zhu and Martinez, 2010 and Kia, Cortes, and Martinez, 2013)**

*The above estimator is cascadable if \(v = 0\).*
For an estimator which satisfies Condition 1 and is asymptotically mean ergodic and cascadable,

\[
\begin{bmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix}
= \begin{bmatrix}
A_0 & 0 \\
\frac{w^T(I - A_0)}{w^T} & 1
\end{bmatrix}
+ \lambda \begin{bmatrix}
A_1 & -A_1p_0 \\
0 & 0
\end{bmatrix}.
\]
For an estimator which satisfies Condition 1 and is asymptotically mean ergodic and cascadable,

\[
\begin{bmatrix}
A(\lambda) & B(\lambda) \\
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A_0 & 0 \\
\frac{w^T(I - A_0)}{\lambda} & 1
\end{bmatrix}
+ \lambda
\begin{bmatrix}
A_1 & -A_1p_0 \\
0 & 0
\end{bmatrix}.
\]

Now choose \(A_0\) and \(A_1\) to minimize the worst-case asymptotic convergence factor subject to Condition 1.
Outline

1. Introduction
2. Motivation: Static consensus
3. Problem Setup
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Van Scy, Freeman, Lynch
Robust Optimization of Spectral Radius

Solve: \[ \alpha = \min_{A_i} \max_{\lambda \in \Lambda} \rho(A(\lambda)) \] subject to Condition 1

Lemma 6 (Henrion et al., 2003)
Let \( \bar{\lambda} \) be fixed. Then \( \rho(A(\bar{\lambda})) < \alpha \) if and only if \( H > 0 \) where \( H \) is a matrix whose coefficients are polynomials in \( A_0 \) and \( A_1 \).

Lemma 7 (Chesi, 2013)
\( \rho(A(\lambda)) < \alpha \) for all \( \lambda \in [\lambda_{\min}, \lambda_{\max}] \) if and only if the following conditions hold:
- \( \rho(A(\bar{\lambda})) < \alpha \) for some \( \bar{\lambda} \in [\lambda_{\min}, \lambda_{\max}] \);
- several matrix inequalities are satisfied which are polynomials in \( A_0 \) and \( A_1 \).
Robust Optimization of Spectral Radius

Solve: \[ \alpha = \min_{A_i} \max_{\lambda \in \Lambda} \rho(A(\lambda)) \text{ subject to Condition 1} \]

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Lemma 6 (Henrion et al., 2003)

Let \( \tilde{\lambda} \) be fixed. Then \( \rho(A(\tilde{\lambda})) < \alpha \) if and only if \( H > 0 \) where \( H \) is a matrix whose coefficients are polynomials in \( A_0 \) and \( A_1 \).

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\[ \rho(A(\lambda)) < \alpha \] for all \( \lambda \in [\lambda_{\text{min}}, \lambda_{\text{max}}] \) if and only if the following conditions hold:

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- several matrix inequalities are satisfied which are polynomials in \( A_0 \) and \( A_1 \).
Solutions to PMIs (Henrion and Lasserre, 2006):

- Solve convex LMI relaxations
- Solution converges to the global optimum as size of relaxation increases
- Finite convergence can be detected
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- Solve convex LMI relaxations
- Solution converges to the global optimum as size of relaxation increases
- Finite convergence can be detected

For $A_0, A_1 \in \mathbb{R}^{2 \times 2}$, the conditions can be expressed as LMI's.
Outline

1 Introduction

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3 Problem Setup

4 Robust Optimization of Spectral Radius

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Problem 1

To satisfy Condition 1, we can use the parameterization

\[ A(\lambda) = \begin{bmatrix} 1 - \gamma & 0 \\ 0 & 1 \end{bmatrix} + \lambda \begin{bmatrix} -k_p & k_l \\ -1 & 0 \end{bmatrix} \]

without loss of generality which gives

\[
\begin{bmatrix} p_1 \\ p_0 - \nu \end{bmatrix} = \begin{bmatrix} 0 \\ -\gamma/k_l \end{bmatrix}, \quad \begin{bmatrix} q_1 \\ q_0 - w \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1/\gamma \\ 0 \end{bmatrix}.
\]

Choose \( \nu = q_0 = 0 \) for the estimator to be asymptotically mean ergodic and cascadable.
Estimator Form

\[
\begin{bmatrix}
A(\lambda) & B(\lambda) \\
C(\lambda) & D(\lambda)
\end{bmatrix}
= \begin{bmatrix}
1 - \gamma & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1
\end{bmatrix} + \lambda \begin{bmatrix}
-k_p & k_I & -k_I \\
-1 & 0 & -1 \\
0 & 0 & 0
\end{bmatrix}
\]

Choose $\gamma$, $k_p$, and $k_I$ to minimize the worst-case asymptotic convergence factor of $A(\lambda)$. 
Motivation: Static consensus

Problem Setup

Robust Optimization of Spectral Radius

Example: 2-state estimator

Graph connectivity ($\lambda_{\text{min}} / \lambda_{\text{max}}$)

Worst-case asymptotic convergence factor ($\alpha$)

smaller is better
Introduction
Motivation: Static consensus
Problem Setup
Robust Optimization of Spectral Radius
Example: 2-state estimator

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Worst-case asymptotic convergence factor ($\alpha$)

Non-robust
Robust

smaller is better

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Optimal Worst-Case Dynamic Average Consensus
In general, 

\[ \alpha = \alpha\left(\frac{\lambda_{\min}}{\lambda_{\max}}\right), \]

\( \alpha \) is monotonically non-increasing.
General design procedure

In general,
- \( \alpha = \alpha(\lambda_{\text{min}}/\lambda_{\text{max}}) \),
- \( \alpha \) is monotonically non-increasing.

General design procedure:
1. Design the weighted Laplacian to maximize the ratio \( \lambda_{\text{min}}/\lambda_{\text{max}} \).
2. Design the estimator to have the desired properties:
   - exact
   - robust to initial conditions
   - asymptotically mean ergodic
   - asymptotic convergence factor
To conclude, we have:

- characterized the structure of estimators which are exact and robust to initial conditions,
- used this structure to minimize the worst-case asymptotic convergence factor for a two-dimensional estimator, and
- setup the problem of designing higher dimensional estimators as a set of PMIs. Techniques exist for finding the global optimal solution, although they are computationally difficult.
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Chesi, Graziano (2013). “Exact robust stability analysis of uncertain systems with a scalar parameter via LMIs”. In: *Automatica* 49.4, pp. 1083 –1086. ISSN: 0005-1098. DOI: http://dx.doi.org/10.1016/j.automatica.2013.01.033.


Peterson, Cameron K. and Derek A. Paley (2013). “Distributed Estimation for Motion Coordination in an Unknown Spatially Varying Flowfield”. In: Journal of Guidance, Control, and Dynamics 36.3, pp. 894–898. ISSN: 0731-5090. DOI: 10.2514/1.59453.


Closed-Form Solution

The LMIs in Lemma 7 can be solved in closed-form to obtain

\[
\alpha = \begin{cases} 
\frac{\lambda_r^2 - 8\lambda_r + 8}{8 - \lambda_r^2}, & 0 < \lambda_r \leq 3 - \sqrt{5} \\
\frac{\sqrt{(1 - \lambda_r)(5\lambda_r^2 - \lambda_r^3 + 4) - \lambda_r(1 - \lambda_r)}}{2(\lambda_r^2 + 1)}, & 3 - \sqrt{5} < \lambda_r \leq 1
\end{cases}
\]

where \( \lambda_r = \lambda_{\text{min}} / \lambda_{\text{max}} \) and the estimator parameters are

\[
\gamma = 1 - \alpha, \quad k_p = \frac{1}{\lambda_{\text{max}}} \frac{\alpha(1 - \alpha)\lambda_r}{\alpha + \lambda_r - 1}, \quad k_I = \frac{1 - \alpha}{\lambda_{\text{min}}}
\]