Feedforward Estimators for the Distributed Average Tracking of Bandlimited Signals in Discrete Time with Switching Graph Topology

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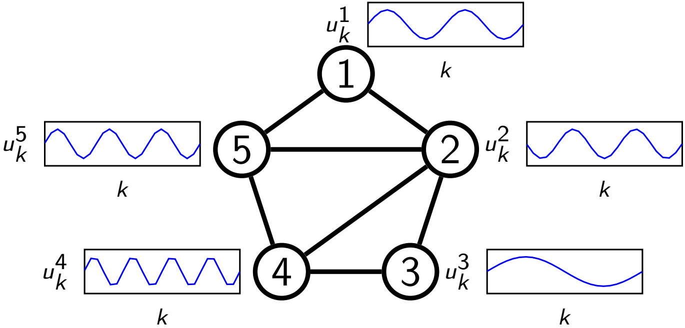
Dec. 13, 2016

Distributed average tracking



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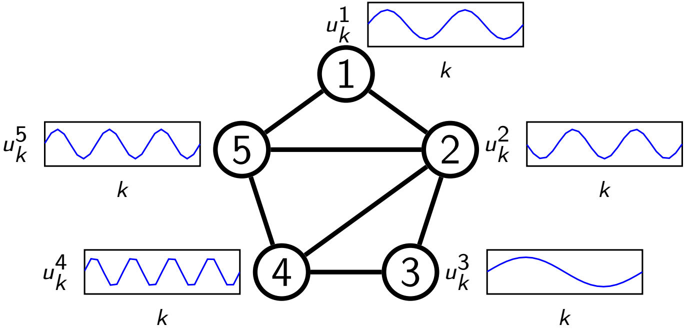
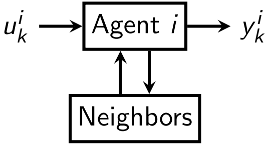




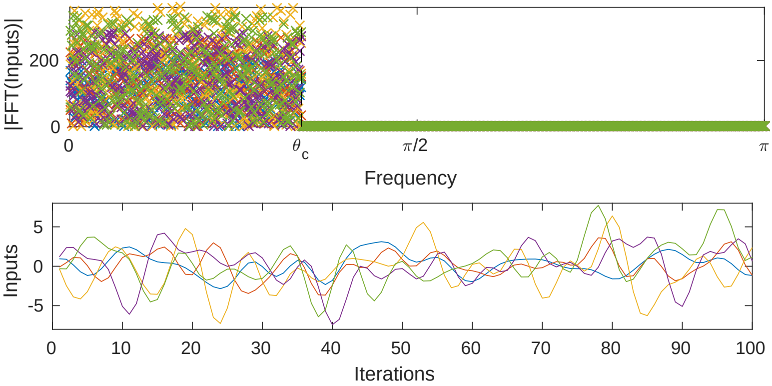
$$e_k = \max_i \left| y_k^i - \frac{1}{N} \sum_{i=1}^{N} u_k^i \right|$$

L_k : Graph Laplacian at time k

Assumptions on the input signals

Assumption (Input signals)

The input signals are bandlimited with cutoff frequency $\theta_c < \pi$ *where* θ_c *is known.*



Assumptions on the graph

Assumption (Graph)

At each iteration k,

- *the graph is connected,*
- *the graph is undirected, and*
- *the nonzero eigenvalues of L_k are in the interval $[\lambda_{min}, \lambda_{max}]$ where λ_{min} and λ_{max} are known.*

Robustness Properties

Definition (Robust to initial conditions)

Initial states do not affect the steady-state.

Definition (Robust to changes in the graph)

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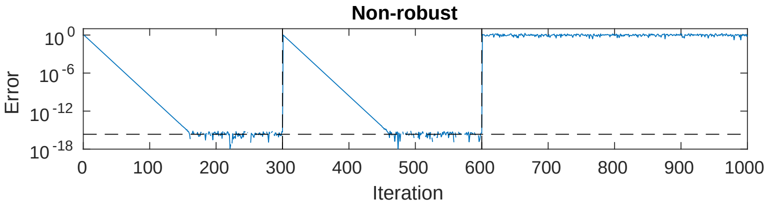
For a given set of graphs,

maximum steady-state error with worst-case constant graph	=	*maximum steady-state error with worst-case sequence of switching graphs*

Example: Robust to changes in the graph

Scenario: The graph changes once at iteration 300 and then at every iteration past 600.



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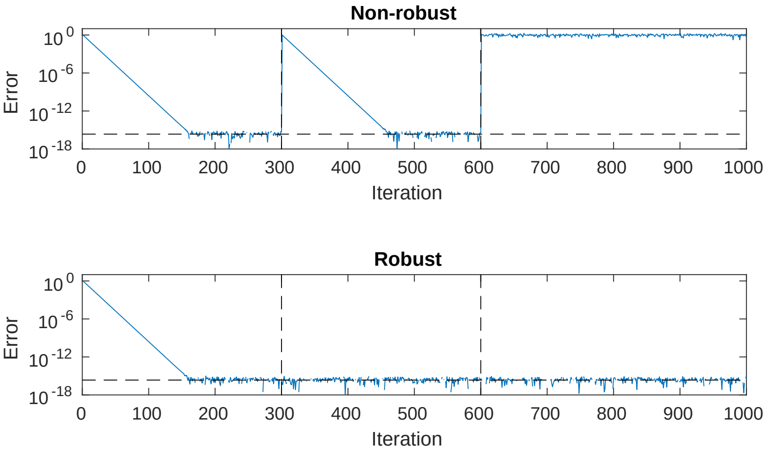

Contribution

We propose an estimator for distributed average tracking which has all of the following properties:

- discrete-time updates
- robust to initial conditions
- robust to changes in the graph
  - proof for undirected graphs
  - simulations for balanced directed graphs
- arbitrarily small steady-state error (using exact arithmetic)

Literature review

Estimator	Discrete-time	Robust to IC's	Switching graphs	Balanced digraphs	Arbitrarily small error
(F. Chen, Y. Cao, W. Ren, 2012)	✗	✗	✓	✓	✓
(M. Zhu, S. Martínez, 2010)	✓	✗	✓	✓	✗
(S. Kia, J. Cortés, S. Martínez, 2013)	✓	✗	✓	✓	✗
(M. Franceschelli, A. Gasparri, 2016)	✓	✓	✓	✓	✗
(B. Van Scoy, R. Freeman, K. Lynch, 2015)	✓	✓	✗	✗	✗
This paper	✓	✓	✓	✓	✓

Static consensus (Tsitsiklis, 1984)

As a first step, consider the following estimator:

$$x_{k+1} = \underbrace{(I - k_p L)}_{W} x_k, \quad x_0 = u$$

Output: $y_k = x_k = (I - k_p L)^k u$

Static consensus: u enters system as initial condition

Issue

Can only track constant inputs.

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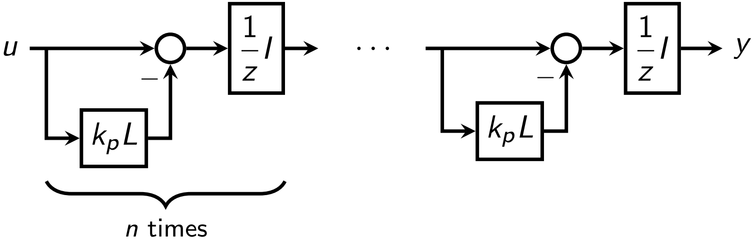
Instead, apply n steps of consensus in a feedforward fashion.



Output: $y_k = (I - k_p L)^n u_{k-n}$

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The output is delayed from the input by n iterations.

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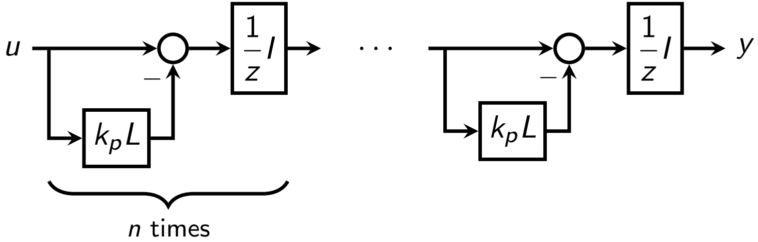

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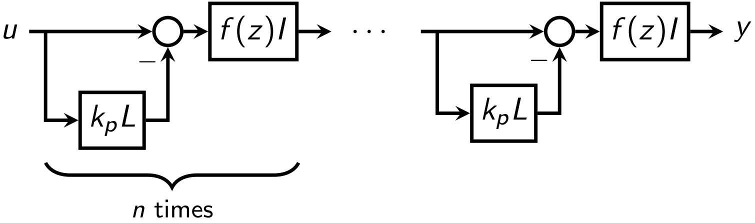
Block Diagram 2

To get rid of the delay in the passband, replace $1/z$ by a filter $f(z)$ where

- f is strictly proper, and
- $f(e^{j\theta}) \approx 1$ for $\theta \in [0, \theta_c]$.



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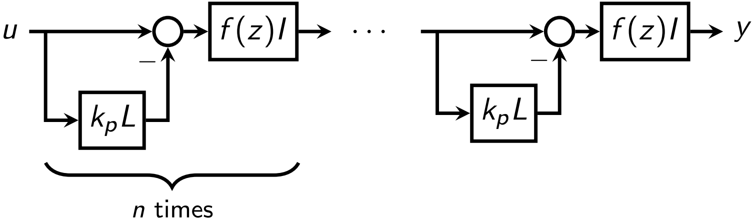

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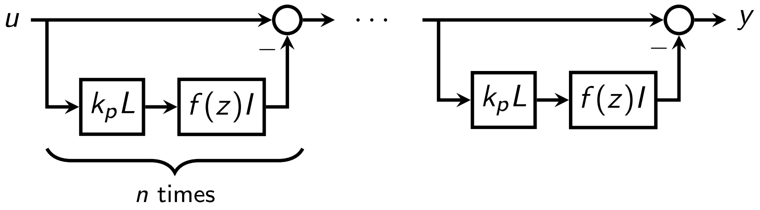
Block Diagram 3

We could instead place $f(z)$ only in the disagreement directions so that the gain in the consensus direction is unity.



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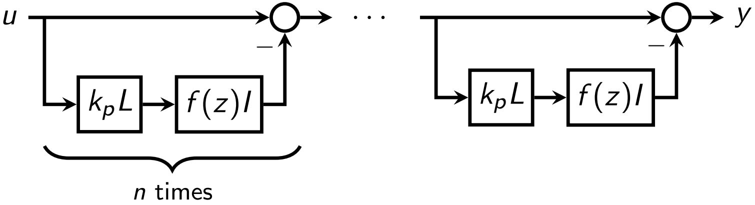

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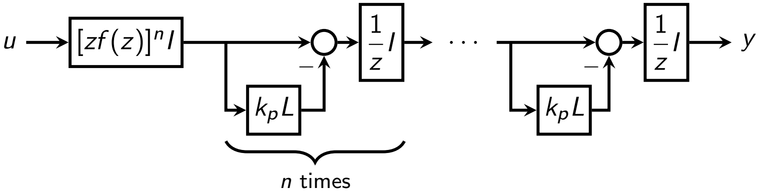
Block Diagram 4

To make the estimator robust to changes in the graph, move $f(z)$ *before* any communication.



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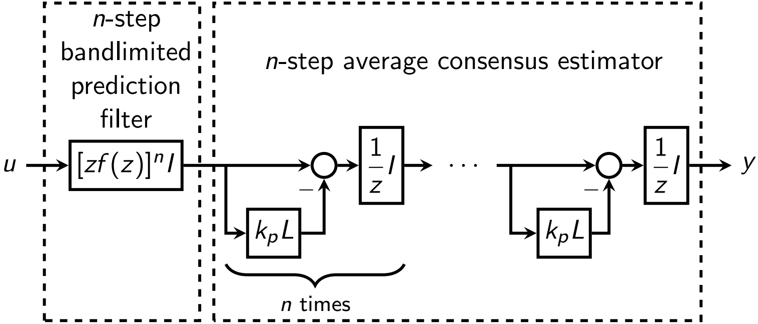

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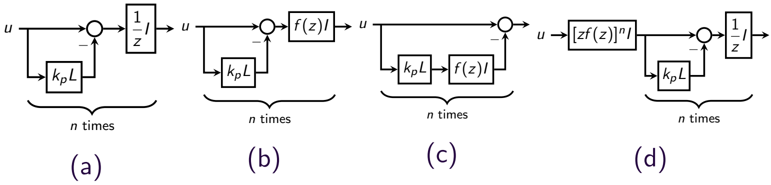
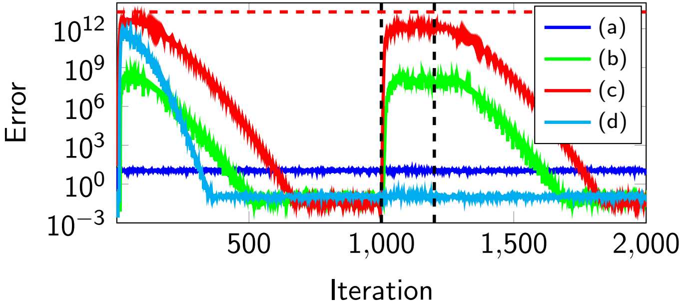
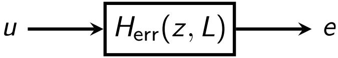
Comparison


u
k_pL
$\frac{1}{z}I$
n times
(a)
u
k_pL
$f(z)I$
n times
(b)
u
k_pL
$f(z)I$
n times
(c)
u
$[zf(z)]^nI$
k_pL
$\frac{1}{z}I$
n times
(d)


Error
10^{12}
10^9
10^6
10^3
10^0
10^{-3}
500
1,000
1,500
2,000
Iteration
(a)
(b)
(c)
(d)

Singular values of the error system



Let $\sigma(\theta, \lambda)$ denote the singular values of the error transfer function. Then the maximum singular value is

$$\sigma_{\text{max}} := \max_{\substack{\lambda \in \{0\} \cup [\lambda_{\text{min}}, \lambda_{\text{max}}] \\ \theta \in [0, \theta_c]}} \sigma(\theta, \lambda).$$

The maximum steady-state error is bounded by

$$||e||_\infty \leq \sigma_{\text{max}} \sqrt{N} ||u||_\infty.$$

Main design problem

Distributed Average Tracking

Van Scoy, Freeman, Lynch

Problem Setup

Block Diagrams

Analysis

Simulations

Problem

Given

- *the cutoff frequency* (θ_c)
- *the Laplacian eigenvalue region* (λ_{min} *and* λ_{max})

choose

- *the number of stages* (n)
- *the filter* $f(z)$

to minimize σ_{max}.

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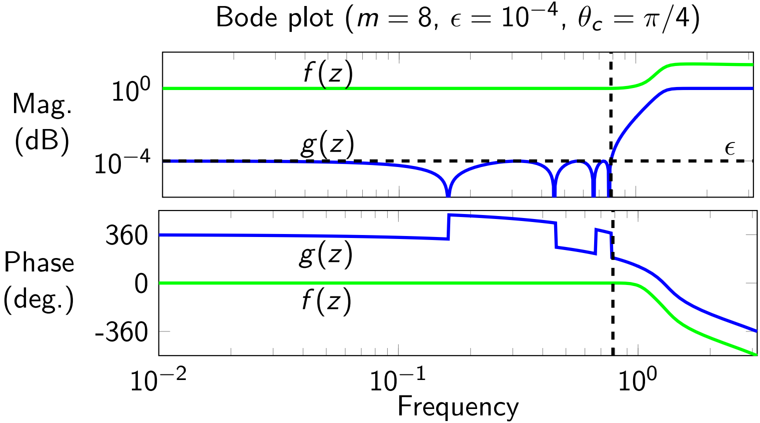
Question

How to design $f(z)$ *such that*

- f *is strictly proper, and*
- $f(e^{j\theta}) \approx 1$ *for* $\theta \in [0, \theta_c]$?

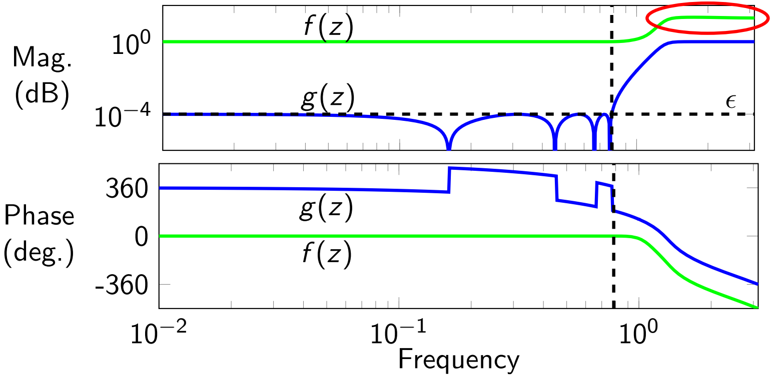
Design of $f(z)$



$$f(z) = 1 - \frac{g(z)}{\lim_{z\to\infty} g(z)}$$

Design of $f(z)$

Bode plot ($m = 8$, $\epsilon = 10^{-4}$, $\theta_c = \pi/4$)



Issue

If $||f||_\infty$ is too large, the estimator is numerically unstable!

Optimization problem

Problem

Given θ_c, λ_{min}, λ_{max}, H_{max}, and m_{max}, solve

$$\min_{n,m,\epsilon} \sigma_{max} \quad s.t. \; ||H||_\infty \leq H_{max}, \quad m \leq m_{max}.$$

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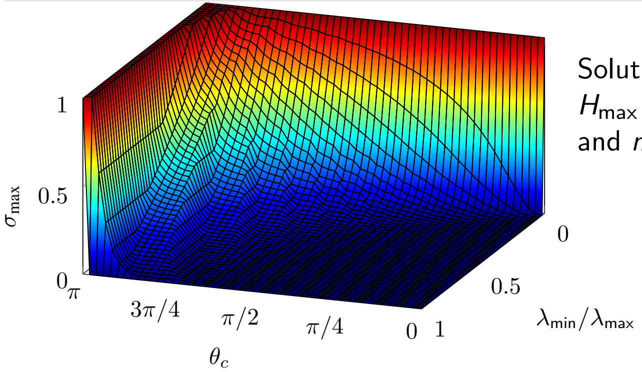


Solution for $H_{\max} = 2 \times 10^{13}$ and $m_{\max} = 8$.

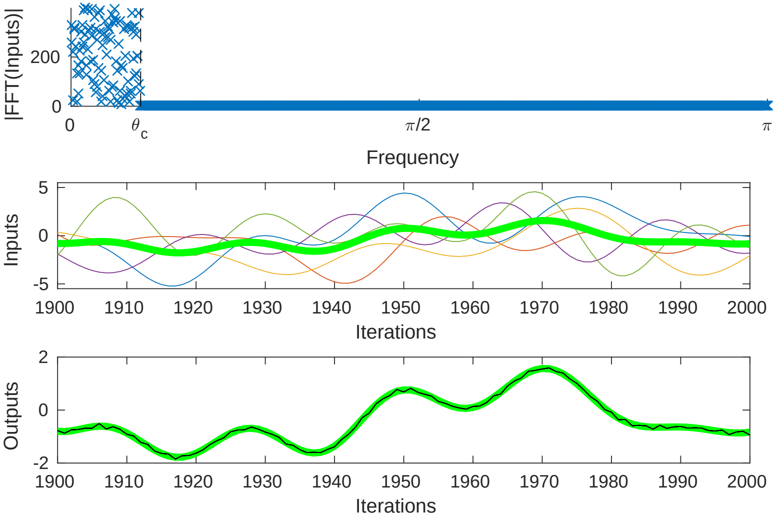
Theorem: Arbitrarily small error

Theorem

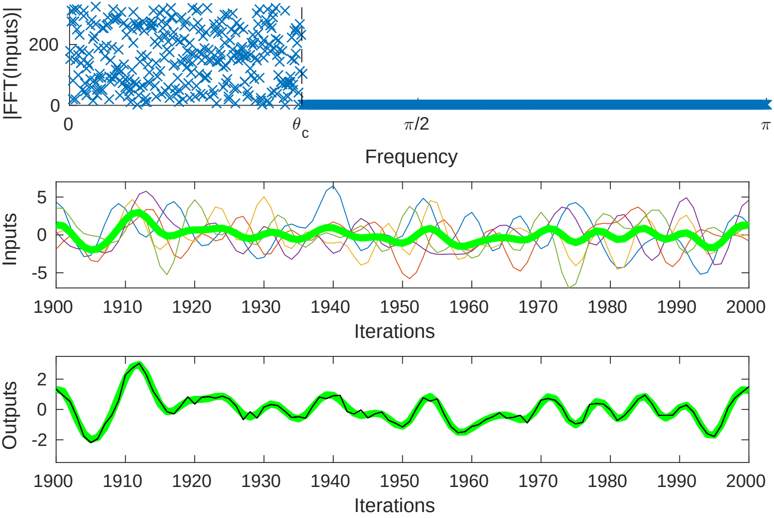
The steady-state error can be made arbitrarily small if

1. *the number of stages is arbitrarily large*
2. *the number of states on each estimator is arbitrarily large*
3. *exact arithmetic is used*

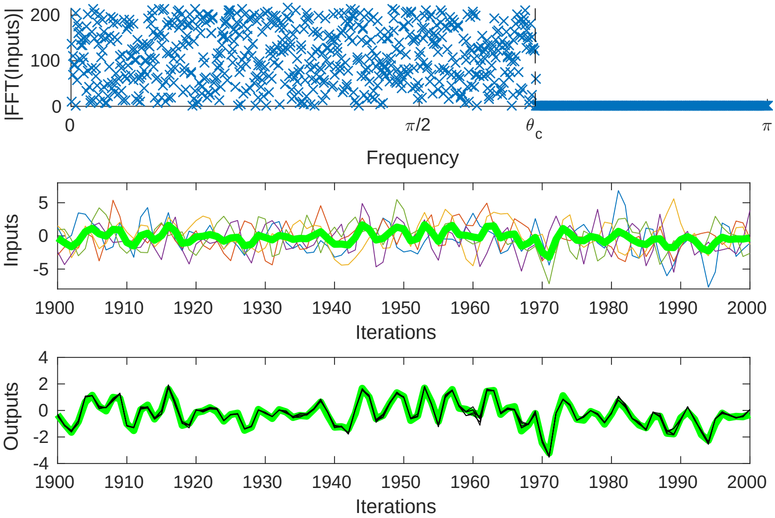
Simulation: $\theta_c = \pi/10$



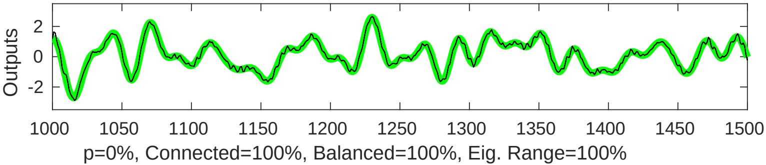
Simulation: $\theta_c = \pi/3$



Simulation: $\theta_c = 2\pi/3$



Dropped packets



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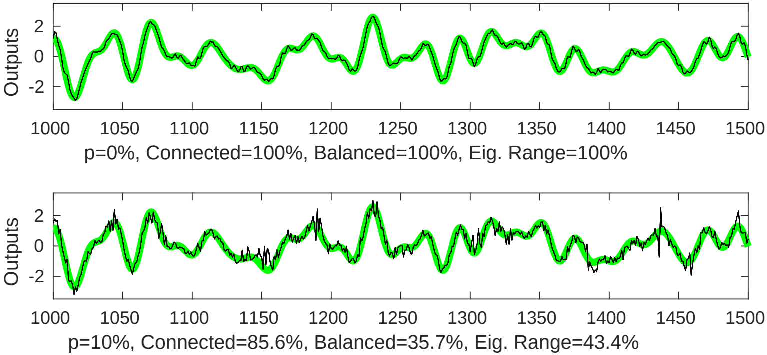

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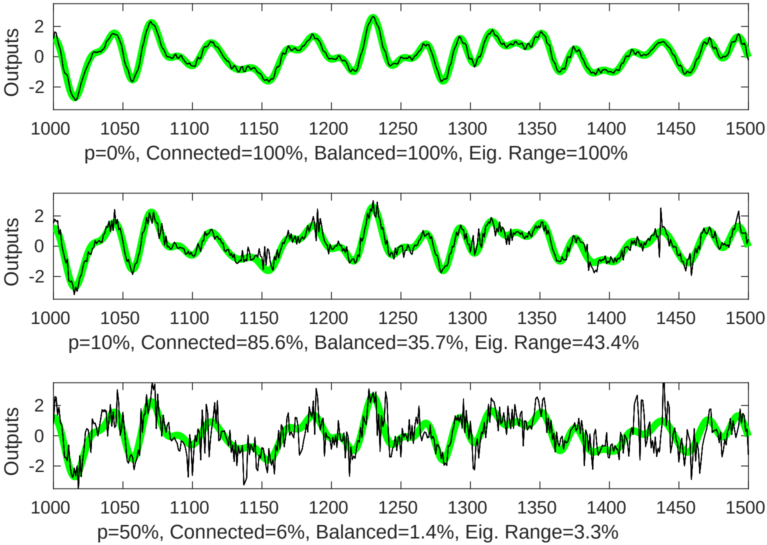

Conclusions

A feedforward estimator is proposed to solve the distributed average tracking problem of bandlimited signals in discrete-time. The estimator has the following properties:

- discrete-time updates
- robust to initial conditions
- robust to changes in the graph
- robust to directed communication (from simulations)
- arbitrarily small steady-state error (using exact arithmetic)