

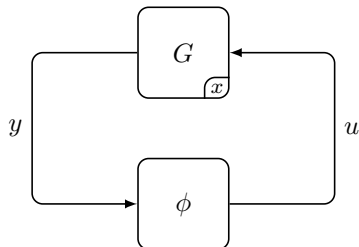
Absolute Stability via Lifting and Interpolation

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Absolute stability

Find conditions on the LTI system G with state x that ensure stability for all initial states and all nonlinearities ϕ in a function class.

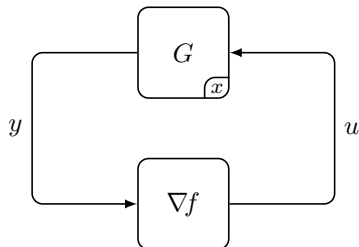


Contributions

- directly construct a Lyapunov function
- simple dissipation proof that naturally generalizes to other settings
- relate the set of all valid multipliers to interpolation

Absolute stability

Find conditions on the LTI system G with state x that ensure stability for all initial states and all nonlinearities ϕ in a function class.



Assumptions (non-restrictive)

- G is SISO
- $f : \mathbb{R} \rightarrow \mathbb{R}$ is convex with $f(0) = 0$ and $\nabla f(0) = 0$

Quadratic constraints

Quadratic inequalities that hold between u and y when $u_t = \nabla f(y_t)$ for some convex f .

- **Willems, Brockett**

$$\begin{bmatrix} u_\ell \\ \vdots \\ u_0 \end{bmatrix}^\top M \begin{bmatrix} y_\ell \\ \vdots \\ y_0 \end{bmatrix} \geq 0 \quad \text{where} \quad M \text{ doubly hyperdominant}$$

- **Zames, Falb, O'Shea**

$$\sum_{t=-\infty}^{\infty} u_t (\Pi y)_t \geq 0 \quad \text{where} \quad \sum_{t=-\infty}^{\infty} \pi_t \geq 0 \text{ and } \pi_t \leq 0 \text{ for } t \neq 0$$

Frequency domain

The system is absolutely stable if there exists a multiplier Π such that

$$\operatorname{Re}\{\Pi(z) G(z)\} < 0$$

for all z on the unit circle.

- follows from the main IQC result (Megretski & Rantzer, 1996)
- frequency domain inequality must hold at an infinite number of points
- tractable search in the time domain over a subset of multipliers

Multiplier factorization

If π has finite duration ℓ , then we can factor the multiplier as

$$\Pi(z) = \Psi(z)^* \begin{bmatrix} 0 & M^T \\ M & 0 \end{bmatrix} \Psi(z)$$

where

$$\Psi(z) = \begin{bmatrix} 1 & 0 \\ z^{-1} & 0 \\ \dots & 0 \\ z^{-\ell} & 0 \\ 0 & 1 \\ 0 & z^{-1} \\ 0 & \dots \\ 0 & z^{-\ell} \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} \pi_0 & \pi_1 & \dots & \pi_\ell \\ \pi_{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-\ell} & 0 & \dots & 0 \end{bmatrix}$$

Use the factorization to define the augmented system with state x_t

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right] = \Psi(z) \begin{bmatrix} G(z) \\ 1 \end{bmatrix}$$

Time domain

Apply the positive real lemma to the FDI to obtain the equivalent LMI

$$P \succ 0$$
$$0 \succ \begin{bmatrix} \mathbf{A}^\top P \mathbf{A} - P & \mathbf{A}^\top P \mathbf{B} \\ \mathbf{B}^\top P \mathbf{A} & \mathbf{B}^\top P \mathbf{B} \end{bmatrix} + \frac{1}{2} [\mathbf{C} \quad \mathbf{D}]^\top \begin{bmatrix} 0 & M^\top \\ M & 0 \end{bmatrix} [\mathbf{C} \quad \mathbf{D}]$$

LMI feasible $\xRightarrow{\text{PR}}$ FDI feasible $\xRightarrow{\text{IQC}}$ absolute stability

Issues

- does not produce a Lyapunov function
- how to construct the multipliers for other function classes?

A simple time-domain proof

Multiply the LMI by (\mathbf{x}_t, u_t) to obtain the dissipation inequality

$$0 > \mathbf{x}_{t+1}^\top P \mathbf{x}_{t+1} - \mathbf{x}_t^\top P \mathbf{x}_t + \begin{bmatrix} u_t \\ u_{t-1} \\ \vdots \\ u_{t-\ell} \end{bmatrix}^\top \begin{bmatrix} \pi_0 & \pi_1 & \dots & \pi_\ell \\ \pi_{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-\ell} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-\ell} \end{bmatrix}$$

Then sum over t from 0 to T

$$0 > \mathbf{x}_{T+1}^\top P \mathbf{x}_{T+1} - \mathbf{x}_0^\top P \mathbf{x}_0 + \begin{bmatrix} u_T \\ u_{T-1} \\ \vdots \\ u_0 \end{bmatrix}^\top \begin{bmatrix} \pi_0 & \pi_1 & \dots & \pi_T \\ \pi_1 & \pi_0 & \dots & \pi_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-T} & \pi_{1-T} & \dots & \pi_0 \end{bmatrix} \begin{bmatrix} y_T \\ y_{T-1} \\ \vdots \\ y_0 \end{bmatrix}$$

From the conditions on the multiplier, the matrix involving π is doubly hyperdominant, so the quadratic form is nonnegative.

A conservative approach

Feasibility of the LMI implies $V(\mathbf{x}) = \mathbf{x}^\top P \mathbf{x}$ is a Lyapunov function.

$$P \succ 0$$

$$0 \prec \begin{bmatrix} \mathbf{A}^\top P \mathbf{A} - P & \mathbf{A}^\top P \mathbf{B} \\ \mathbf{B}^\top P \mathbf{A} & \mathbf{B}^\top P \mathbf{B} \end{bmatrix} + \frac{1}{2} [\mathbf{C} \quad \mathbf{D}]^\top \begin{bmatrix} 0 & M^\top \\ M & 0 \end{bmatrix} [\mathbf{C} \quad \mathbf{D}]$$

M doubly hyperdominant

But this approach is very conservative!

Main idea

- lift the iterates to a higher-dimensional space

$$\mathbf{y}_t = \begin{bmatrix} y_t \\ \vdots \\ y_{t-\ell} \end{bmatrix} \quad \mathbf{u}_t = \begin{bmatrix} u_t \\ \vdots \\ u_{t-\ell} \end{bmatrix} \quad \mathbf{f}_t = \begin{bmatrix} f_t \\ \vdots \\ f_{t-\ell} \end{bmatrix}$$

- use interpolation to find all quadratic-plus-linear inequalities

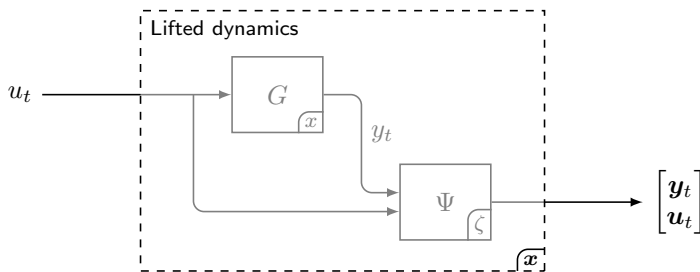
$$\mathbf{y}_t^\top M \mathbf{u}_t + m^\top \mathbf{f}_t \geq 0$$

- use the inequalities to search for a common quadratic Lyapunov function in the lifted space

$$V(\mathbf{x}_t, \mathbf{f}_t) = \mathbf{x}_t^\top P \mathbf{x}_t + p^\top \mathbf{f}_t$$

This approach constructs a Lyapunov function and recovers the best known results.

Lifted system



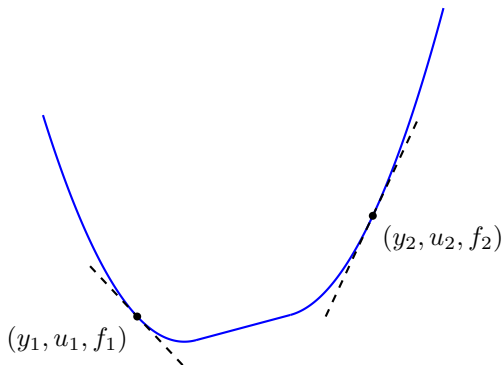
$$\begin{bmatrix} \mathbf{x}_{t+1} \\ \mathbf{y}_t \\ \mathbf{u}_t \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x}_t \\ \mathbf{u}_t \end{bmatrix} \quad \text{and} \quad \mathbf{F} \mathbf{f}_{t+1} = \mathbf{F}_+ \mathbf{f}_t$$

where

$$\left[\begin{array}{c|c} \mathbf{A} & \mathbf{B} \\ \hline \mathbf{C} & \mathbf{D} \end{array} \right] = \Psi(z) \begin{bmatrix} G(z) \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{aligned} \mathbf{F} &= \begin{bmatrix} 0_{\ell \times 1} & I_{\ell} \end{bmatrix} \\ \mathbf{F}_+ &= \begin{bmatrix} I_{\ell} & 0_{\ell \times 1} \end{bmatrix} \end{aligned}$$

Convex interpolation

When does there exist a convex function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(0) = 0$ and $\nabla f(0) = 0$ such that $u_i = \nabla f(y_i)$ and $f_i = f(y_i)$?



Necessary and sufficient conditions (Taylor, Hendrickx, Glineur, 2017)

$$f_i \geq f_j + u_j^\top (y_i - y_j), \quad u_i^\top y_i \geq f_i, \quad f_i \geq 0 \quad \text{for all } i, j$$

Interpolation cone

$$\mathcal{K} = \left\{ \left(\begin{bmatrix} y_\ell^\top u_\ell & \dots & y_\ell^\top u_1 \\ \vdots & \ddots & \vdots \\ y_1^\top u_\ell & \dots & y_1^\top u_1 \end{bmatrix}, \begin{bmatrix} f_\ell \\ \vdots \\ f_1 \end{bmatrix} \right) \mid (y_i, u_i, f_i) \text{ is interpolable} \right\}$$

For convex functions,

$$\mathcal{K} = \{(\mathbf{G}, \mathbf{f}) \mid f_i \geq f_j + \mathbf{G}_{ij} - \mathbf{G}_{jj}, \mathbf{G}_{ii} \geq f_i, \\ \text{and } f_i \geq 0 \text{ for all } i, j \text{ and } \text{rank}(\mathbf{G}) = 1\}$$

The interpolation cone \mathcal{K} characterizes the set of Gramians \mathbf{G} and function values \mathbf{f} that are interpolable.

Quadratic-plus-linear inequalities

$$\mathbf{y}_t^T M \mathbf{u}_t + m^T \mathbf{f}_t \geq 0$$

This holds for all multipliers (M, m) in the dual of the interpolation cone.

$$\mathcal{K}^* = \{(M, m) \mid \text{tr}(M^T \mathbf{G}) + m^T \mathbf{f} \geq 0 \text{ for all } (\mathbf{G}, \mathbf{f}) \in \mathcal{K}\}$$

For convex functions,

$$\mathcal{K}^* = \{(M, m) \mid M^T \mathbf{1} \geq 0, M \mathbf{1} + m \geq 0, \text{ and } M_{ij} \leq 0 \text{ for all } i \neq j\}$$

The set of all multipliers is the dual of the interpolation cone.

Lyapunov function

$$V(\mathbf{x}, \mathbf{f}) = \mathbf{x}^\top P \mathbf{x} + p^\top \mathbf{f}$$

- Dissipation inequality

$$V(\mathbf{x}_{t+1}, \mathbf{f}_{t+1}) - V(\mathbf{x}_t, \mathbf{f}_t) + \sigma_1(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) \leq 0$$

- Positivity

$$\|\mathbf{x}_t\|^2 - V(\mathbf{x}_t, \mathbf{f}_t) + \sigma_2(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) \leq 0$$

- Multipliers

$$\sigma_i(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) = \mathbf{y}_t^\top M_i \mathbf{u}_t + m_i^\top \mathbf{f}_t \quad (M_i, m_i) \in \mathcal{K}^*$$

Proof: $\|\mathbf{x}_t\|^2 \leq V(\mathbf{x}_t, \mathbf{f}_t) \leq V(\mathbf{x}_{t-1}, \mathbf{f}_{t-1}) \leq \dots \leq V(\mathbf{x}_0, \mathbf{f}_0)$

Main result

$$\begin{aligned} \begin{bmatrix} \mathbf{A}^\top P \mathbf{A} - P & \mathbf{A}^\top P \mathbf{B} \\ \mathbf{B}^\top P \mathbf{A} & \mathbf{B}^\top P \mathbf{B} \end{bmatrix} + \frac{1}{2} [\mathbf{C} \quad \mathbf{D}]^\top \begin{bmatrix} 0 & M_1^\top \\ M_1 & 0 \end{bmatrix} [\mathbf{C} \quad \mathbf{D}] \preceq 0 \\ (\mathbf{F}_+ - \mathbf{F})^\top p + m_1 \leq 0 \\ \begin{bmatrix} I - P & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} [\mathbf{C} \quad \mathbf{D}]^\top \begin{bmatrix} 0 & M_2^\top \\ M_2 & 0 \end{bmatrix} [\mathbf{C} \quad \mathbf{D}] \preceq 0 \\ -\mathbf{F}^\top p + m_2 \leq 0 \end{aligned}$$

- symmetric matrix P
- vector p
- multipliers (M_1, m_1) and (M_2, m_2) in the dual cone \mathcal{K}^*

Feasibility of the LMI implies that $V(\mathbf{x}, \mathbf{f})$ is a Lyapunov function.

Numerical examples

- $G(z)$ in negative feedback with slope-restricted nonlinearity in $(0, \alpha)$
- find the largest α for which the system is absolutely stable

Ex.	Plant $G(z)$	α	(n_b, n_f)	ℓ
1	$\frac{0.1z}{z^2 - 1.8z + 0.81}$	12.9960	(1, 0)	1
2	$\frac{z^3 - 1.95z^2 + 0.9z + 0.05}{z^4 - 2.8z^3 + 3.5z^2 - 2.412z + 0.7209}$	0.8027	(1, 4)	4
3	$-\frac{z^3 - 1.95z^2 + 0.9z + 0.05}{z^4 - 2.8z^3 + 3.5z^2 - 2.412z + 0.7209}$	0.3054	(0, 1)	1
4	$\frac{z^4 - 1.5z^3 + 0.5z^2 - 0.5z + 0.5}{4.4z^5 - 8.957z^4 + 9.893z^3 - 5.671z^2 + 2.207z - 0.5}$	3.8240	(0, 4)	4
5	$\frac{-0.5z + 0.1}{z^3 - 0.9z^2 + 0.79z + 0.089}$	2.4475	(0, 1)	1
6	$\frac{2z + 0.92}{z^2 - 0.5z}$	0.9114	(1, 2)	2
7	$\frac{1.341z^4 - 1.221z^3 + 0.6285z^2 - 0.5618z + 0.1993}{z^5 - 0.935z^4 + 0.7697z^3 - 1.118z^2 + 0.6917z - 0.1352}$	0.4347	(3, 3)	3

Lyapunov function for Example 6

$$V(\mathbf{x}_t, \mathbf{f}_t) = \begin{bmatrix} x_{t-2} \\ u_{t-2} \\ u_{t-1} \end{bmatrix}^T P \begin{bmatrix} x_{t-2} \\ u_{t-2} \\ u_{t-1} \end{bmatrix} + p^T \begin{bmatrix} f_{t-1} \\ f_{t-2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1.4483 & -0.2173 & -2.4073 & -2.4262 \\ -0.2173 & 0.8523 & -2.6369 & 0.1214 \\ -2.4073 & -2.6369 & 2.4142 & -1.5938 \\ -2.4262 & 0.1214 & -1.5938 & 0.4756 \end{bmatrix} \quad p = \begin{bmatrix} -6.1534 \\ -3.2837 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 8.6813 & -8.6813 & -0.0000 \\ -0.0000 & 5.8115 & -5.8115 \\ -2.5025 & -0.0000 & 2.5278 \end{bmatrix} \quad m_1 = \begin{bmatrix} -6.1788 \\ 2.8698 \\ 3.2837 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 11.2412 & -3.3521 & -1.6564 \\ -1.6595 & 10.6892 & -2.7451 \\ -1.3351 & -1.5047 & 5.9290 \end{bmatrix} \quad m_2 = \begin{bmatrix} -8.2467 \\ -5.8325 \\ -0.0000 \end{bmatrix}$$

Extensions

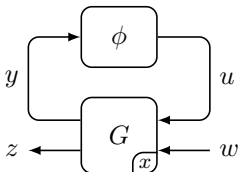
- continuous time

$$0 \geq \frac{d}{dt} V(\mathbf{x}(t)) + \sigma(\mathbf{y}(t), \mathbf{u}(t), \mathbf{f}(t))$$

- exponential stability

$$0 \geq V(\mathbf{x}_{t+1}, \mathbf{f}_{t+1}) - \rho^2 V(\mathbf{x}_t, \mathbf{f}_t) + \sigma(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t)$$

Robust quadratic performance



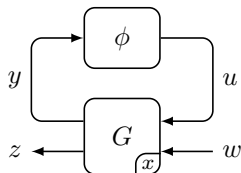
For all exogenous input signals w ,

$$\begin{aligned} V(\mathbf{x}_{t+1}, \mathbf{f}_{t+1}) - V(\mathbf{x}_t, \mathbf{f}_t) + \sigma_1(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) &\leq \sigma_p(w_t, z_t) \\ \|\mathbf{x}_t\|^2 - V(\mathbf{x}_t, \mathbf{f}_t) + \sigma_2(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) &\leq 0 \end{aligned}$$

$$\sum_{t=0}^T \sigma_p(w_t, z_t) \geq 0$$

For example, if $\sigma_p := \gamma^2 \|w_t\|^2 - \|z_t\|^2$, then G has a robust ℓ_2 gain from $w \rightarrow z$ of γ .

Robust stochastic performance



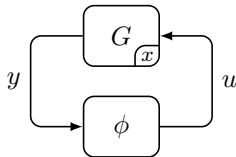
Suppose w_t is i.i.d. zero-mean random noise with covariance Σ .

$$\begin{aligned} V(\mathbf{x}_{t+1}, \mathbf{f}_{t+1}) - V(\mathbf{x}_t, \mathbf{f}_t) + \sigma_1(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) + \|z_t\|^2 &\leq 0 \\ -V(\mathbf{x}_t, \mathbf{f}_t) + \sigma_2(\mathbf{y}_t, \mathbf{u}_t, \mathbf{f}_t) &\leq 0 \\ \text{tr}(P\mathbf{B}_w\Sigma\mathbf{B}_w^\top) &\leq \gamma^2 \end{aligned}$$

$$\limsup_{T \rightarrow \infty} \mathbb{E} \left[\frac{1}{T} \sum_{t=0}^{T-1} \|z_t\|^2 \right] \leq \gamma^2$$

Approach

- lift the iterates to a higher-dimensional space
- use interpolation to find all quadratic-plus-linear inequalities
- search for a common quadratic Lyapunov function in the lifted space



Benefits

- directly construct a Lyapunov function
- simple dissipation proof that naturally generalizes to other settings
- relate the set of all valid multipliers to interpolation