Absolute Stability via Lifting and Interpolation

Bryan Van Scoy
Miami University

Laurent Lessard
Northeastern University

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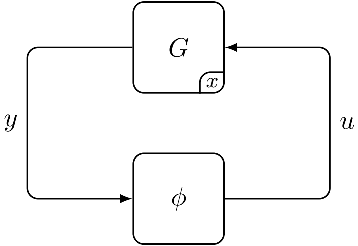
Absolute stability

Find conditions on the LTI system G with state x that ensure stability for all initial states and all nonlinearities ϕ in a function class.

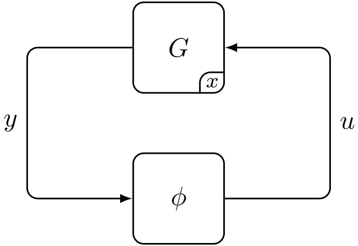


Contributions

- directly construct a Lyapunov function
- simple dissipation proof that naturally generalizes to other settings
- relate the set of all valid multipliers to interpolation

Absolute stability

Find conditions on the LTI system G with state x that ensure stability for all initial states and all nonlinearities ϕ in a function class.



Assumptions

- G is SISO
- $\phi : \mathbb{R} \to \mathbb{R}$ is the gradient of a convex function and $\phi(0) = 0$

Quadratic constraints

Quadratic inequalities that hold between u and y when $u_t = \phi(y_t)$ for some ϕ that is the gradient of a convex function.

- **Willems, Brockett**

$$\begin{bmatrix} u_\ell \\ \vdots \\ u_0 \end{bmatrix}^\mathsf{T} M \begin{bmatrix} y_\ell \\ \vdots \\ y_0 \end{bmatrix} \geq 0 \qquad \text{where} \qquad M \text{ doubly hyperdominant}$$

- **Zames, Falb, O'Shea**

$$\sum_{t=-\infty}^{\infty} u_t \, (\Pi y)_t \geq 0 \qquad \text{where} \qquad \sum_{t=-\infty}^{\infty} \pi_t \geq 0 \text{ and } \pi_t \leq 0 \text{ for } t \neq 0$$

Frequency domain

The system is absolutely stable if there exists a multiplier Π such that

$$\text{Re}\{\Pi(z)\, G(z)\} < 0$$

for all z on the unit circle.

- follows from the main IQC result (Megretski & Rantzer, 1996)
- frequency domain inequality must hold at an infinite number of points
- tractable search in the time domain over a subset of multipliers

Multiplier factorization

If π has finite duration ℓ, then we can factor the multiplier as

$$\Pi(z) = \Psi(z)^* \begin{bmatrix} 0 & M^\mathsf{T} \\ M & 0 \end{bmatrix} \Psi(z)$$

where

$$\Psi(z) = \begin{bmatrix} 1 & 0 \\ z^{-1} & 0 \\ \dots & 0 \\ z^{-\ell} & 0 \\ 0 & 1 \\ 0 & z^{-1} \\ 0 & \dots \\ 0 & z^{-\ell} \end{bmatrix} \qquad \text{and} \qquad M = \begin{bmatrix} \pi_0 & \pi_1 & \dots & \pi_\ell \\ \pi_{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-\ell} & 0 & \dots & 0 \end{bmatrix}$$

Use the factorization to define the augmented system with state $\boldsymbol{x}_t$

$$\left[\begin{array}{c|c} \boldsymbol{A} & \boldsymbol{B} \\ \hline \boldsymbol{C} & \boldsymbol{D} \end{array}\right] = \Psi(z) \begin{bmatrix} G(z) \\ 1 \end{bmatrix}$$

Time domain

Apply the positive real lemma to the FDI to obtain the equivalent LMI

$$P \succ 0$$

$$0 \succ \begin{bmatrix} \boldsymbol{A}^\mathsf{T} P \boldsymbol{A} - P & \boldsymbol{A}^\mathsf{T} P \boldsymbol{B} \\ \boldsymbol{B}^\mathsf{T} P \boldsymbol{A} & \boldsymbol{B}^\mathsf{T} P \boldsymbol{B} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & M^\mathsf{T} \\ M & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}$$

$$\text{LMI feasible} \quad \overset{\text{PR}}{\Longrightarrow} \quad \text{FDI feasible} \quad \overset{\text{IQC}}{\Longrightarrow} \quad \text{absolute stability}$$

Issues

- does not produce a Lyapunov function
- how to construct the multipliers for other function classes?

A simple time-domain proof

Multiply the LMI by $(\boldsymbol{x}_t, u_t)$ to obtain the dissipation inequality

$$0 > \boldsymbol{x}_{t+1}^\mathsf{T} P \boldsymbol{x}_{t+1} - \boldsymbol{x}_t^\mathsf{T} P \boldsymbol{x}_t + \begin{bmatrix} u_t \\ u_{t-1} \\ \vdots \\ u_{t-\ell} \end{bmatrix}^\mathsf{T} \begin{bmatrix} \pi_0 & \pi_1 & \dots & \pi_\ell \\ \pi_{-1} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-\ell} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} y_t \\ y_{t-1} \\ \vdots \\ y_{t-\ell} \end{bmatrix}$$

Then sum over t from 0 to T

$$0 > \boldsymbol{x}_{T+1}^\mathsf{T} P \boldsymbol{x}_{T+1} - \boldsymbol{x}_0^\mathsf{T} P \boldsymbol{x}_0 + \begin{bmatrix} u_T \\ u_{T-1} \\ \vdots \\ u_0 \end{bmatrix}^\mathsf{T} \begin{bmatrix} \pi_0 & \pi_1 & \dots & \pi_T \\ \pi_1 & \pi_0 & \dots & \pi_{T-1} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{-T} & \pi_{1-T} & \dots & \pi_0 \end{bmatrix} \begin{bmatrix} y_T \\ y_{T-1} \\ \vdots \\ y_0 \end{bmatrix}$$

From the conditions on the multiplier, the matrix involving π is doubly hyperdominant, so the quadratic form is nonnegative.

A conservative approach

Feasibility of the LMI implies $V(\boldsymbol{x}) = \boldsymbol{x}^\mathsf{T} P \boldsymbol{x}$ is a Lyapunov function.

$$
\begin{aligned}
&P \succ 0 \\
&0 \succ \begin{bmatrix} \boldsymbol{A}^\mathsf{T} P \boldsymbol{A} - P & \boldsymbol{A}^\mathsf{T} P \boldsymbol{B} \\ \boldsymbol{B}^\mathsf{T} P \boldsymbol{A} & \boldsymbol{B}^\mathsf{T} P \boldsymbol{B} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & M^\mathsf{T} \\ M & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \\
&M \text{ doubly hyperdominant}
\end{aligned}
$$

But this approach is very conservative!

Main idea

- lift the iterates to a higher-dimensional space

$$\boldsymbol{y}_t = \begin{bmatrix} y_t \\ \vdots \\ y_{t-\ell} \end{bmatrix} \qquad \boldsymbol{u}_t = \begin{bmatrix} u_t \\ \vdots \\ u_{t-\ell} \end{bmatrix} \qquad \boldsymbol{f}_t = \begin{bmatrix} f_t \\ \vdots \\ f_{t-\ell} \end{bmatrix}$$

- use interpolation to find all quadratic-plus-linear inequalities

$$\boldsymbol{y}_t^\mathsf{T} M \, \boldsymbol{u}_t + m^\mathsf{T} \boldsymbol{f}_t \geq 0$$

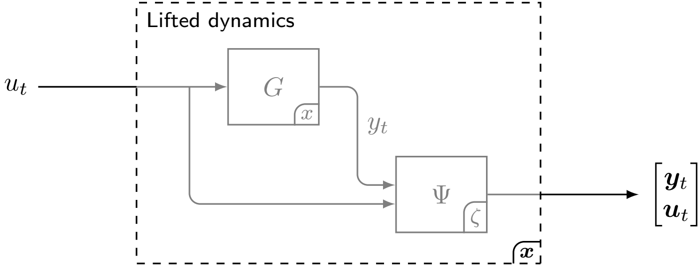
- use the inequalities to search for a common quadratic Lyapunov function in the lifted space

$$V(\boldsymbol{x}_t, \boldsymbol{f}_t) = \boldsymbol{x}_t^\mathsf{T} P \, \boldsymbol{x}_t + p^\mathsf{T} \boldsymbol{f}_t$$

This approach constructs a Lyapunov function
and recovers the best known results.

The iterates satisfy $f_t = f(y_t)$ and $u_t = \nabla f(y_t)$, where the nonlinearity is $\phi = \nabla f$.

Lifted system



$$\begin{bmatrix}\boldsymbol{x}_{t+1}\\ \boldsymbol{y}_t\\ \boldsymbol{u}_t\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{x}_t\\ u_t\end{bmatrix} \qquad \text{and} \qquad \boldsymbol{F}\boldsymbol{f}_{t+1} = \boldsymbol{F}_+\boldsymbol{f}_t$$

where

$$\left[\begin{array}{c|c}\boldsymbol{A} & \boldsymbol{B}\\ \hline \boldsymbol{C} & \boldsymbol{D}\end{array}\right] = \Psi(z)\begin{bmatrix}G(z)\\ 1\end{bmatrix} \qquad \text{and} \qquad \begin{aligned}\boldsymbol{F} &= \begin{bmatrix}0_{\ell\times 1} & I_\ell\end{bmatrix}\\ \boldsymbol{F}_+ &= \begin{bmatrix}I_\ell & 0_{\ell\times 1}\end{bmatrix}\end{aligned}$$

Convex interpolation

When does there exist a convex function $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 0$ and $\nabla f(0) = 0$ such that $u_i = \nabla f(y_i)$ and $f_i = f(y_i)$?

Necessary and sufficient conditions from (Taylor, Hendrickx, Glineur, 2017)

$$f_i \geq f_j + u_j^\mathsf{T}(y_i - y_j), \quad u_i^\mathsf{T} y_i \geq f_i, \quad f_i \geq 0 \quad \text{for all } i, j$$

This is equivalent to $(\boldsymbol{y}\boldsymbol{u}^\mathsf{T}, \boldsymbol{f}) \in \mathcal{K}$, where $\mathcal{K}$ is the convex cone

$$\mathcal{K} = \{(\boldsymbol{G}, \boldsymbol{f}) \mid f_i \geq f_j + G_{ij} - G_{jj},\ G_{ii} \geq f_i, \\ \text{and } f_i \geq 0 \text{ for all } i, j \text{ and } \operatorname{rank}(\boldsymbol{G}) = 1\}$$

The interpolation cone $\mathcal{K}$ characterizes the set of Gramians $\boldsymbol{G}$ and function values $\boldsymbol{f}$ that are interpolable.

Quadratic-plus-linear inequalities

$$\boldsymbol{y}_t^{\mathsf{T}} M \boldsymbol{u}_t + m^{\mathsf{T}} \boldsymbol{f}_t \geq 0$$

This holds for all multipliers (M, m) in the dual of the interpolation cone.

$$\mathcal{K}^* = \left\{(M, m) \;\middle|\; \operatorname{tr}(M^{\mathsf{T}} \boldsymbol{G}) + m^{\mathsf{T}} \boldsymbol{f} \geq 0 \text{ for all } (\boldsymbol{G}, \boldsymbol{f}) \in \mathcal{K}\right\}$$

For convex functions,

$$\mathcal{K}^* = \{(M, m) \mid M^{\mathsf{T}} \mathbf{1} \geq 0,\ M\mathbf{1} + m \geq 0,\ \text{and } M_{ij} \leq 0 \text{ for all } i \neq j\}$$

The set of all multipliers is the dual of the interpolation cone.

Lyapunov function

$$V(\boldsymbol{x}, \boldsymbol{f}) = \boldsymbol{x}^\mathsf{T} P \, \boldsymbol{x} + p^\mathsf{T} \boldsymbol{f}$$

- Dissipation inequality

$$V(\boldsymbol{x}_{t+1}, \boldsymbol{f}_{t+1}) - V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma_1(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) \le 0$$

- Positivity

$$\|\boldsymbol{x}_t\|^2 - V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma_2(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) \le 0$$

- Multipliers

$$\sigma_i(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) = \boldsymbol{y}_t^\mathsf{T} M_i \boldsymbol{u}_t + m_i^\mathsf{T} \boldsymbol{f}_t \qquad (M_i, m_i) \in \mathcal{K}^*$$

Proof: $\|\boldsymbol{x}_t\|^2 \le V(\boldsymbol{x}_t, \boldsymbol{f}_t) \le V(\boldsymbol{x}_{t-1}, \boldsymbol{f}_{t-1}) \le \ldots \le V(\boldsymbol{x}_0, \boldsymbol{f}_0)$

Main result

$$\begin{bmatrix} \boldsymbol{A}^\mathsf{T} P \boldsymbol{A} - P & \boldsymbol{A}^\mathsf{T} P \boldsymbol{B} \\ \boldsymbol{B}^\mathsf{T} P \boldsymbol{A} & \boldsymbol{B}^\mathsf{T} P \boldsymbol{B} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & M_1^\mathsf{T} \\ M_1 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \preceq 0$$

$$(\boldsymbol{F}_+ - \boldsymbol{F})^\mathsf{T} p + m_1 \le 0$$

$$\begin{bmatrix} I - P & 0 \\ 0 & 0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & M_2^\mathsf{T} \\ M_2 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \boldsymbol{D} \end{bmatrix} \preceq 0$$

$$-\boldsymbol{F}^\mathsf{T} p + m_2 \le 0$$

- symmetric matrix P
- vector p
- multipliers (M_1, m_1) and (M_2, m_2) in the dual cone $\mathcal{K}^*$

Feasibility of the LMI implies that $V(\boldsymbol{x}, \boldsymbol{f})$ is a Lyapunov function.

Numerical examples

- $G(z)$ in negative feedback with slope-restricted nonlinearity in $(0, \alpha)$
- find the largest α for which the system is absolutely stable

Ex.	Plant $G(z)$	α	(n_b, n_f)	ℓ
1	$\frac{0.1z}{z^2-1.8z+0.81}$	12.9960	$(1,0)$	1
2	$\frac{z^3-1.95z^2+0.9z+0.05}{z^4-2.8z^3+3.5z^2-2.412z+0.7209}$	0.8027	$(1,4)$	4
3	$-\frac{z^3-1.95z^2+0.9z+0.05}{z^4-2.8z^3+3.5z^2-2.412z+0.7209}$	0.3054	$(0,1)$	1
4	$\frac{z^4-1.5z^3+0.5z^2-0.5z+0.5}{4.4z^5-8.957z^4+9.893z^3-5.671z^2+2.207z-0.5}$	3.8240	$(0,4)$	4
5	$\frac{-0.5z+0.1}{z^3-0.9z^2+0.79z+0.089}$	2.4475	$(0,1)$	1
6	$\frac{2z+0.92}{z^2-0.5z}$	0.9114	$(1,2)$	2
7	$\frac{1.341z^4-1.221z^3+0.6285z^2-0.5618z+0.1993}{z^5-0.935z^4+0.7697z^3-1.118z^2+0.6917z-0.1352}$	0.4347	$(3,3)$	3

Equivalent to $-(1+\alpha G)$ in positive feedback with the gradient of a convex function.

Lyapunov function for Example 6

$$V(\boldsymbol{x}_t, \boldsymbol{f}_t) = \begin{bmatrix} x_{t-2} \\ u_{t-2} \\ u_{t-1} \end{bmatrix}^{\mathsf{T}} P \begin{bmatrix} x_{t-2} \\ u_{t-2} \\ u_{t-1} \end{bmatrix} + p^{\mathsf{T}} \begin{bmatrix} f_{t-1} \\ f_{t-2} \end{bmatrix}$$

$$P = \begin{bmatrix} 1.4483 & -0.2173 & -2.4073 & -2.4262 \\ -0.2173 & 0.8523 & -2.6369 & 0.1214 \\ -2.4073 & -2.6369 & 2.4142 & -1.5938 \\ -2.4262 & 0.1214 & -1.5938 & 0.4756 \end{bmatrix} \qquad p = \begin{bmatrix} -6.1534 \\ -3.2837 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} 8.6813 & -8.6813 & -0.0000 \\ -0.0000 & 5.8115 & -5.8115 \\ -2.5025 & -0.0000 & 2.5278 \end{bmatrix} \qquad m_1 = \begin{bmatrix} -6.1788 \\ 2.8698 \\ 3.2837 \end{bmatrix}$$

$$M_2 = \begin{bmatrix} 11.2412 & -3.3521 & -1.6564 \\ -1.6595 & 10.6892 & -2.7451 \\ -1.3351 & -1.5047 & 5.9290 \end{bmatrix} \qquad m_2 = \begin{bmatrix} -8.2467 \\ -5.8325 \\ -0.0000 \end{bmatrix}$$

Extensions

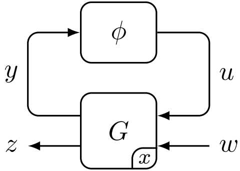
- continuous time

$$0 \geq \frac{\mathsf{d}}{\mathsf{d}t} V(\boldsymbol{x}(t)) + \sigma(\boldsymbol{y}(t), \boldsymbol{u}(t), \boldsymbol{f}(t))$$

- exponential stability

$$0 \geq V(\boldsymbol{x}_{t+1}, \boldsymbol{f}_{t+1}) - \rho^2 \, V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t)$$

Robust quadratic performance



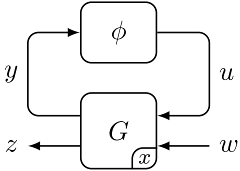
For all exogenous input signals w,

$$V(\boldsymbol{x}_{t+1}, \boldsymbol{f}_{t+1}) - V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma_1(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) \leq \sigma_p(w_t, z_t)$$
$$\|\boldsymbol{x}_t\|^2 - V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma_2(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) \leq 0$$

$$\sum_{t=0}^{T} \sigma_p(w_t, z_t) \geq 0$$

For example, if $\sigma_p := \gamma^2\|w_t\|^2 - \|z_t\|^2$, then G has a robust ℓ_2 gain from $w \to z$ of γ.

Robust stochastic performance



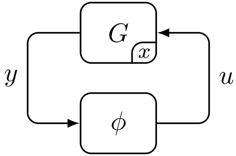
Suppose w_t is i.i.d. zero-mean random noise with covariance Σ.

$$
\begin{aligned}
V(\boldsymbol{x}_{t+1}, \boldsymbol{f}_{t+1}) - V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma_1(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) + \|z_t\|^2 &\le 0 \\
-V(\boldsymbol{x}_t, \boldsymbol{f}_t) + \sigma_2(\boldsymbol{y}_t, \boldsymbol{u}_t, \boldsymbol{f}_t) &\le 0 \\
\operatorname{tr}\left(P \boldsymbol{B}_w \Sigma \boldsymbol{B}_w^\mathsf{T}\right) &\le \gamma^2
\end{aligned}
$$

$$
\limsup_{T\to\infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} \|z_t\|^2\right] \le \gamma^2
$$

Approach

- lift the iterates to a higher-dimensional space
- use interpolation to find all quadratic-plus-linear inequalities
- search for a common quadratic Lyapunov function in the lifted space



Benefits

- directly construct a Lyapunov function
- simple dissipation proof that naturally generalizes to other settings
- relate the set of all valid multipliers to interpolation