# Lyapunov-based approach to the analysis of iterative optimization algorithms

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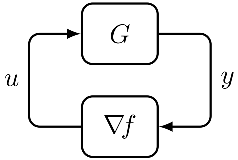
# Algorithm analysis




Represents a gradient-based iterative optimization algorithm. e.g.,

$$x_{k+1} = x_k + \nabla f(x_k) + \beta(x_k - x_{k-1})$$

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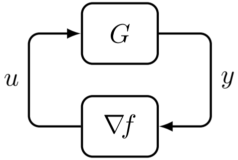


Represents a gradient-based iterative optimization algorithm. e.g.,

$$x_{k+1} = x_k + \nabla f(x_k) + \beta(x_k - x_{k-1})$$

has the following equations:

$$\begin{bmatrix} x_{k+1} \\ x_k \end{bmatrix} = \begin{bmatrix} 1+\beta & -\beta \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix} + \begin{bmatrix} -\alpha \\ 0 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ x_{k-1} \end{bmatrix}$$

$$u_k = \nabla f(y_k)$$

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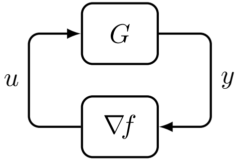


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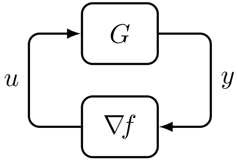
$$x_{k+1} = x_k + \nabla f(x_k) + \beta(x_k - x_{k-1})$$

in state-space notation:

$$y = \left[\begin{array}{cc|c} 1+\beta & -\beta & -\alpha \\ 1 & 0 & 0 \\ \hline 1 & 0 & 0 \end{array}\right] u$$

$$u = \nabla f(y)$$

# Algorithm analysis




Algorithm analysis:

Find conditions on the system $G$ that guarantees certain algorithm performance for all initial conditions $x_0$ and all functions $f$ in a given class.

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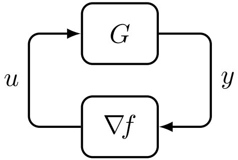


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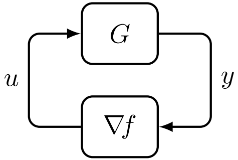
Find conditions on the system $G$ that guarantees certain algorithm performance for all initial conditions $x_0$ and all functions $f$ in a given class.

This is a Lur'e problem!

# Algorithm analysis




**In this tutorial:**

- directly construct a Lyapunov function to certify robust performance
- simple dissipation proof that naturally generalizes to other settings

# Simple example

Iterative algorithm:

$$\left\{\begin{aligned} x_{k+1} &= Ax_k + Bu_k \\ \begin{bmatrix} y_k \\ u_k \end{bmatrix} &= Cx_k + Du_k \end{aligned}\right\} \qquad \text{and} \qquad u_k = \phi(y_k)$$

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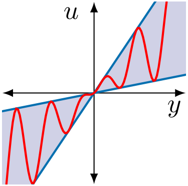
Sector-bounded nonlinearity:



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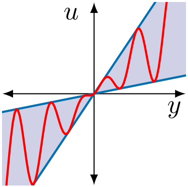
$$my_k \le u_k \le Ly_k$$



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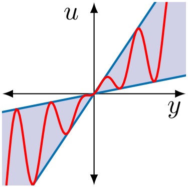

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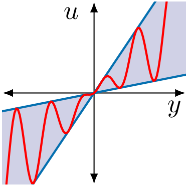
$$\begin{bmatrix} y_k \\ u_k \end{bmatrix}^\mathsf{T} \underbrace{\begin{bmatrix} -2mL & m+L \\ m+L & -2 \end{bmatrix}}_{\Pi} \begin{bmatrix} y_k \\ u_k \end{bmatrix} \ge 0$$



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Sector-bounded nonlinearity:

$$m y_k \le u_k \le L y_k$$

$$(u_k - m y_k)(L y_k - u_k) \ge 0$$

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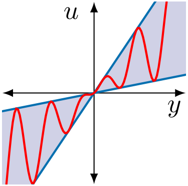


Find a $P \succ 0$ so that for all $\{x_k, u_k, y_k\}$ satisfying the dynamics, the following *dissipation inequality* holds:

$$x_{k+1}^\mathsf{T} P x_{k+1} - \rho^2 x_k^\mathsf{T} P x_k + \begin{bmatrix} y_k \\ u_k \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} y_k \\ u_k \end{bmatrix} \le 0$$

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Therefore: $\quad x_{k+1}^\mathsf{T} P x_{k+1} \le \rho^2 x_k^\mathsf{T} P x_k \quad$ and $\quad x_k^\mathsf{T} P x_k \ge 0$.

So $V(x) = x^\mathsf{T} P x$ is a Lyapunov function that certifies exponential stability.

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Substitute $x_{k+1} = A x_k + B u_k$ and $\begin{bmatrix} y_k \\ u_k \end{bmatrix} = C x_k + D u_k$ and obtain:

$$\begin{bmatrix} x_k \\ u_k \end{bmatrix}^\mathsf{T} \left( \begin{bmatrix} A^\mathsf{T} P A - \rho^2 P & A^\mathsf{T} P B \\ B^\mathsf{T} P A & B^\mathsf{T} P B \end{bmatrix} + \begin{bmatrix} C & D \end{bmatrix}^\mathsf{T} \Pi \begin{bmatrix} C & D \end{bmatrix} \right) \begin{bmatrix} x_k \\ u_k \end{bmatrix} \le 0$$

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Alternative form:

$$\begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix}^\mathsf{T} \begin{bmatrix} P & 0 & 0 \\ 0 & -\rho^2 P & 0 \\ 0 & 0 & \Pi \end{bmatrix} \begin{bmatrix} A & B \\ I & 0 \\ C & D \end{bmatrix} \preceq 0$$

This is a *linear matrix inequality* (LMI) in $P \succ 0$.

We can use a bisection search to find the smallest feasible $\rho$.

**Note:** This is also known as the S-procedure, or the time-domain version of the circle criterion

# Simple example

$$x_{k+1}^{\mathsf{T}} P x_{k+1} - \rho^2 x_k^{\mathsf{T}} P x_k + \begin{bmatrix} y_k \\ u_k \end{bmatrix}^{\mathsf{T}} \Pi \begin{bmatrix} y_k \\ u_k \end{bmatrix} \leq 0$$

with dynamics $x_{k+1} = Ax_k + Bu_k$ and $\begin{bmatrix} y_k \\ u_k \end{bmatrix} = Cx_k + Du_k$.

**Implementation trick:** use row vectors to define *basis elements*. Then, express all other elements in terms of the basis.

```
% define basis (independent rows)
n = size(A,1);
x = [eye(n) zeros(n,1)];
u = [zeros(1,n) 1];

% algorithm dynamics
x1 = A*x + B*u;
yu = C*x + D*u;

% Lyapunov equation (generates LMI)
Pi = [-2*m*L m+L; m+L -2];
x1'*P*x1 - rho^2*x'*P*x + yu'*Pi*yu <= 0
```

# What's next?

- Using *lifting* to represent a more complicated function class
- Using different dissipation inequalities to change performance measure
- Numerical examples

# Smooth strongly convex functions

Consider functions $f : \mathbb{R}^n \to \mathbb{R}$ that

- are $m$-strongly convex:

$$f(x) - \frac{m}{2}\|x\|^2 \text{ is convex}$$

- have $L$-Lipschitz gradients:

$$\|\nabla f(x) - \nabla f(y)\| \leq L\|x - y\| \quad \text{for all } x, y \in \mathbb{R}^n$$

# Interpolation conditions

From [Taylor, Hendrickx, Glineur, 2017]:

Consider $\{(y_k, u_k, f_k)\}$ for $k = 1, \ldots, m$. The following are equivalent.

**a)** There exists a smooth strongly convex $f$ satisfying

$$f(y_k) = f_k \quad \text{and} \quad \nabla f(y_k) = u_k \quad \text{for } k = 1, \ldots, m.$$

**b)** The following inequality holds for $i, j \in \{1, \ldots, m\}$.

$$2(L-m)(f_i - f_j) - mL\|y_i - y_j\|^2 \\ + 2(y_i - y_j)^\mathsf{T}(mu_i - Lu_j) - \|u_i - u_j\|^2 \geq 0.$$

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**b)** The following inequality holds for $i, j \in \{1, \dots, m\}$.

$$\begin{bmatrix} y_i \\ y_j \\ u_i \\ u_j \end{bmatrix}^\mathsf{T} \begin{bmatrix} -mL & mL & m & -L \\ mL & -mL & -m & L \\ m & -m & -1 & 1 \\ -L & L & 1 & -1 \end{bmatrix} \begin{bmatrix} y_i \\ y_j \\ u_i \\ u_j \end{bmatrix} + 2(L-m) \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} f_i \\ f_j \end{bmatrix} \ge 0$$

Involves *pairs of points*, and includes *function values*.

# Interpolation conditions

Lifted iterates:

$$\boldsymbol{y}_k = \begin{bmatrix} y_k \\ \vdots \\ y_{k-\ell} \end{bmatrix} \qquad \boldsymbol{u}_k = \begin{bmatrix} u_k \\ \vdots \\ u_{k-\ell} \end{bmatrix} \qquad \boldsymbol{f}_k = \begin{bmatrix} f_k \\ \vdots \\ f_{k-\ell} \end{bmatrix}$$

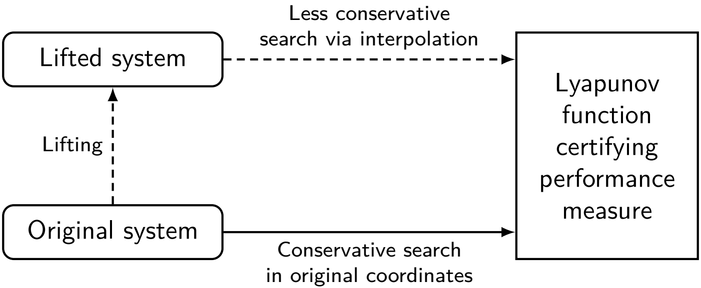
Interpolation conditions

$$\begin{bmatrix} y_i \\ y_j \\ u_i \\ u_j \end{bmatrix}^\mathsf{T} \begin{bmatrix} -mL & mL & m & -L \\ mL & -mL & -m & L \\ m & -m & -1 & 1 \\ -L & L & 1 & -1 \end{bmatrix} \begin{bmatrix} y_i \\ y_j \\ u_i \\ u_j \end{bmatrix} + 2(L-m) \begin{bmatrix} 1 \\ -1 \end{bmatrix}^\mathsf{T} \begin{bmatrix} f_i \\ f_j \end{bmatrix} \geq 0$$

Linear combination with $\Lambda_{ij} \geq 0$ of each interpolation condition:

$$\underbrace{\begin{bmatrix} \boldsymbol{y}_k \\ \boldsymbol{u}_k \end{bmatrix}^\mathsf{T} \Pi(\Lambda) \begin{bmatrix} \boldsymbol{y}_k \\ \boldsymbol{u}_k \end{bmatrix} + \pi(\Lambda)^\mathsf{T} \boldsymbol{f}_k}_{\sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda)} \geq 0$$

# Lifting approach



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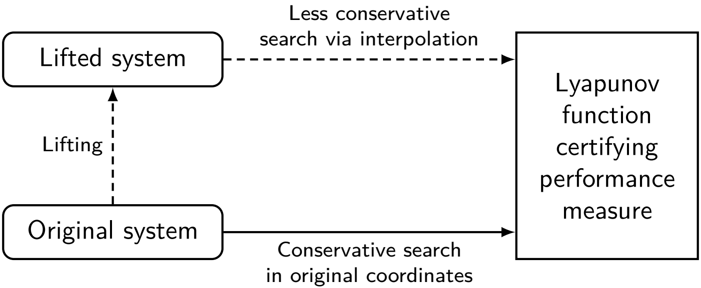
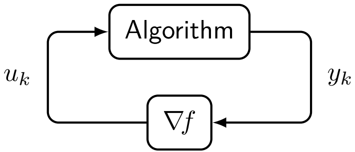



Lifting increases the number of variables but allows use of a simpler Lyapunov function.

**Original system:**




state: $x_k$

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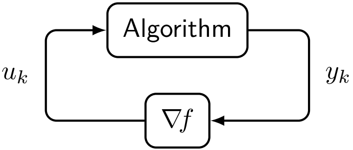


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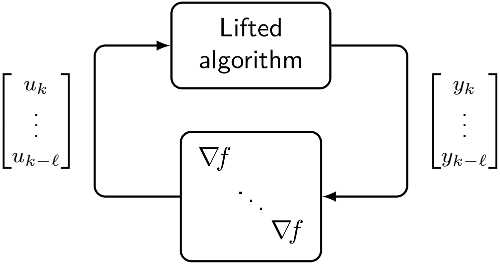
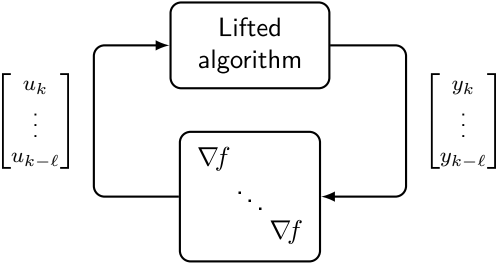
**Lifted system:**




state: $\boldsymbol{x}_k = \begin{bmatrix} x_k \\ y_{k-1} \\ \vdots \\ y_{k-\ell} \\ u_{k-1} \\ \vdots \\ u_{k-\ell} \end{bmatrix}$

# Lifted dynamics




$$\begin{bmatrix} \boldsymbol{x}_{k+1} \\ \boldsymbol{y}_k \\ \boldsymbol{u}_k \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{x}_k \\ u_k \end{bmatrix} \quad \text{and} \quad \mathbf{F} \boldsymbol{f}_{k+1} = \mathbf{F}_+ \boldsymbol{f}_k$$

# Lyapunov function

$$V(\boldsymbol{x}, \boldsymbol{f}) = \boldsymbol{x}^\mathsf{T} P\, \boldsymbol{x} + p^\mathsf{T} \boldsymbol{f}$$

- Dissipation inequality

$$V(\boldsymbol{x}_{k+1}, \boldsymbol{f}_{k+1}) - \rho^2 V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_1) \leq 0$$

- Positivity

$$\|\boldsymbol{x}_k\|^2 - V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_2) \leq 0$$

- Interpolation conditions

$$\sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda) = \begin{bmatrix} \boldsymbol{y}_k \\ \boldsymbol{u}_k \end{bmatrix}^\mathsf{T} \Pi(\Lambda) \begin{bmatrix} \boldsymbol{y}_k \\ \boldsymbol{u}_k \end{bmatrix} + \pi(\Lambda)^\mathsf{T} \boldsymbol{f}_k$$

Proof: $\|\boldsymbol{x}_k\|^2 \leq V(\boldsymbol{x}_k, \boldsymbol{f}_k) \leq \rho^2 V(\boldsymbol{x}_{k-1}, \boldsymbol{f}_{k-1}) \leq \cdots \leq \rho^{2k} V(\boldsymbol{x}_0, \boldsymbol{f}_0)$

# Linear matrix inequality

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ I & 0 \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^\mathsf{T} \begin{bmatrix} P & 0 & 0 \\ 0 & -\rho^2 P & 0 \\ 0 & 0 & \Pi(\Lambda_1) \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ I & 0 \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \preceq 0$$

$$(\mathbf{F}_+ - \rho^2 \mathbf{F})^\mathsf{T} p + \pi(\Lambda_1) \leq 0$$

$$\begin{bmatrix} I & 0 \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^\mathsf{T} \begin{bmatrix} -P & 0 \\ 0 & \Pi(\Lambda_2) \end{bmatrix} \begin{bmatrix} I & 0 \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \preceq 0$$

$$-\mathbf{F}^\mathsf{T} p + \Pi(\Lambda_2) \leq 0$$

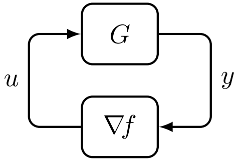
Decision variables:

- symmetric matrix $P$
- vector $p$
- nonnegative coefficients $\Lambda_1$ and $\Lambda_2$.

Feasibility of the LMI implies that $V(\boldsymbol{x}, \boldsymbol{f})$ is a Lyapunov function.

# Beyond convergence rate

# Certifying convergence rate

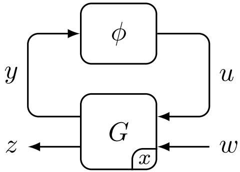



Dissipation inequality and positivity requirement:

$$\begin{aligned} V(\boldsymbol{x}_{k+1}, \boldsymbol{f}_{k+1}) - \rho^2 V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_1) \leq 0 \\ \|\boldsymbol{x}_k\|^2 - V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_2) \leq 0 \end{aligned}$$

$$V(\boldsymbol{x}_{k+1}, \boldsymbol{f}_{k+1}) \leq \rho^2 V(\boldsymbol{x}_k, \boldsymbol{f}_k)$$

# Robust quadratic performance



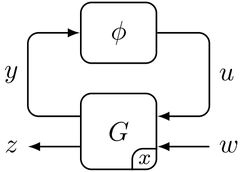

For all exogenous input signals $w$,

$$V(\boldsymbol{x}_{k+1}, \boldsymbol{f}_{k+1}) - V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_1) \le \sigma_p(w_k, z_k)$$
$$\|\boldsymbol{x}_k\|^2 - V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_2) \le 0$$

$$\sum_{t=0}^{T} \sigma_p(w_k, z_k) \ge 0$$

For example, if $\sigma_p := \gamma^2 \|w_k\|^2 - \|z_k\|^2$, then $G$ has a robust $\ell_2$ gain from $w \to z$ of $\gamma$.

# Robust stochastic performance

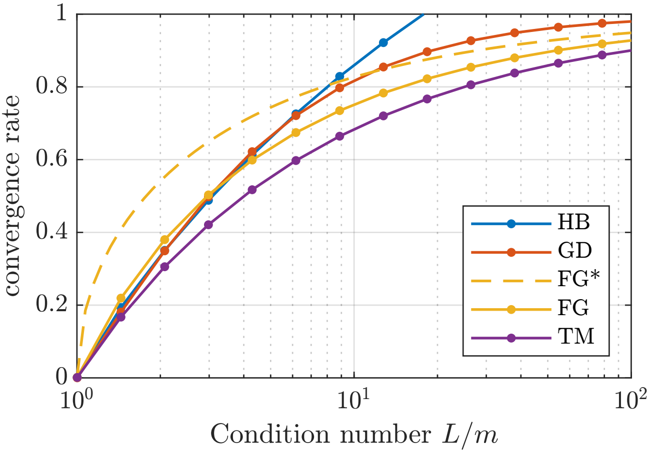



Suppose $w_k$ is i.i.d. zero-mean random noise with covariance $\Sigma$.

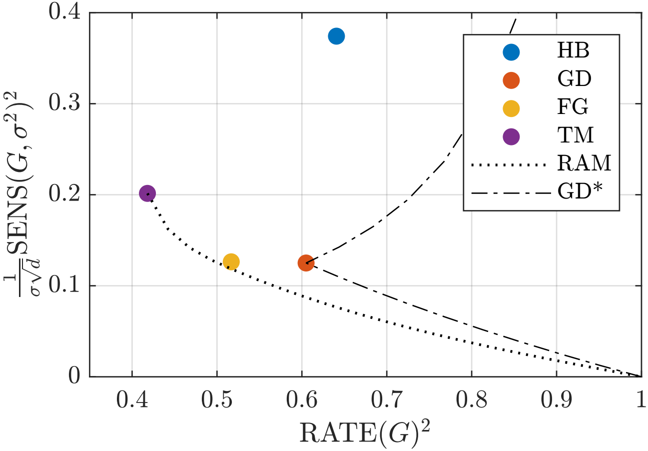
$$
\begin{aligned}
V(\boldsymbol{x}_{k+1}, \boldsymbol{f}_{k+1}) - V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_1) + \|z_k\|^2 &\leq 0 \\
-V(\boldsymbol{x}_k, \boldsymbol{f}_k) + \sigma(\boldsymbol{y}_k, \boldsymbol{u}_k, \boldsymbol{f}_k, \Lambda_2) &\leq 0 \\
\operatorname{tr}\left(P\mathbf{B}_w \Sigma \mathbf{B}_w^{\mathsf{T}}\right) &\leq \gamma^2
\end{aligned}
$$

$$
\limsup_{T\to\infty} \mathbf{E}\left[\frac{1}{T}\sum_{t=0}^{T-1} \|z_k\|^2\right] \leq \gamma^2
$$

# Simulation examples




- Heavy Ball, Gradient Descent, Fast Gradient, Triple Momentum
- Identical to result obtained via IQCs [Lessard, Packard, Recht 2016], [Michalowsky, Scherer, Ebenbauer, 2021]
- Only requires $\ell = 1$.




- Robustness with respect to additive gradient noise
- Lifting approach [Van Scoy, Lessard, 2021] obtains same trade-off curve as IQCs [Michalowsky, Scherer, Ebenbauer, 2021]
- Requires $\ell = 6$.

# Comparison with IQC approach

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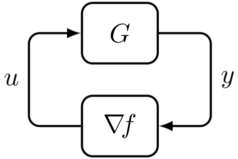
- Lyapunov certificate is more complicated (more degrees of freedom) than IQC certificate

## Approach

- lift the iterates to a higher-dimensional space
- use interpolation to find all valid inequalities
- search for "quadratic + linear" Lyapunov function in the lifted space



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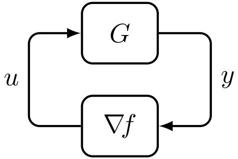


**Benefits**

- directly construct a Lyapunov function
- simple dissipation proof that naturally generalizes to other settings
- idea also extends beyond smooth strongly convex functions

**Thank you**